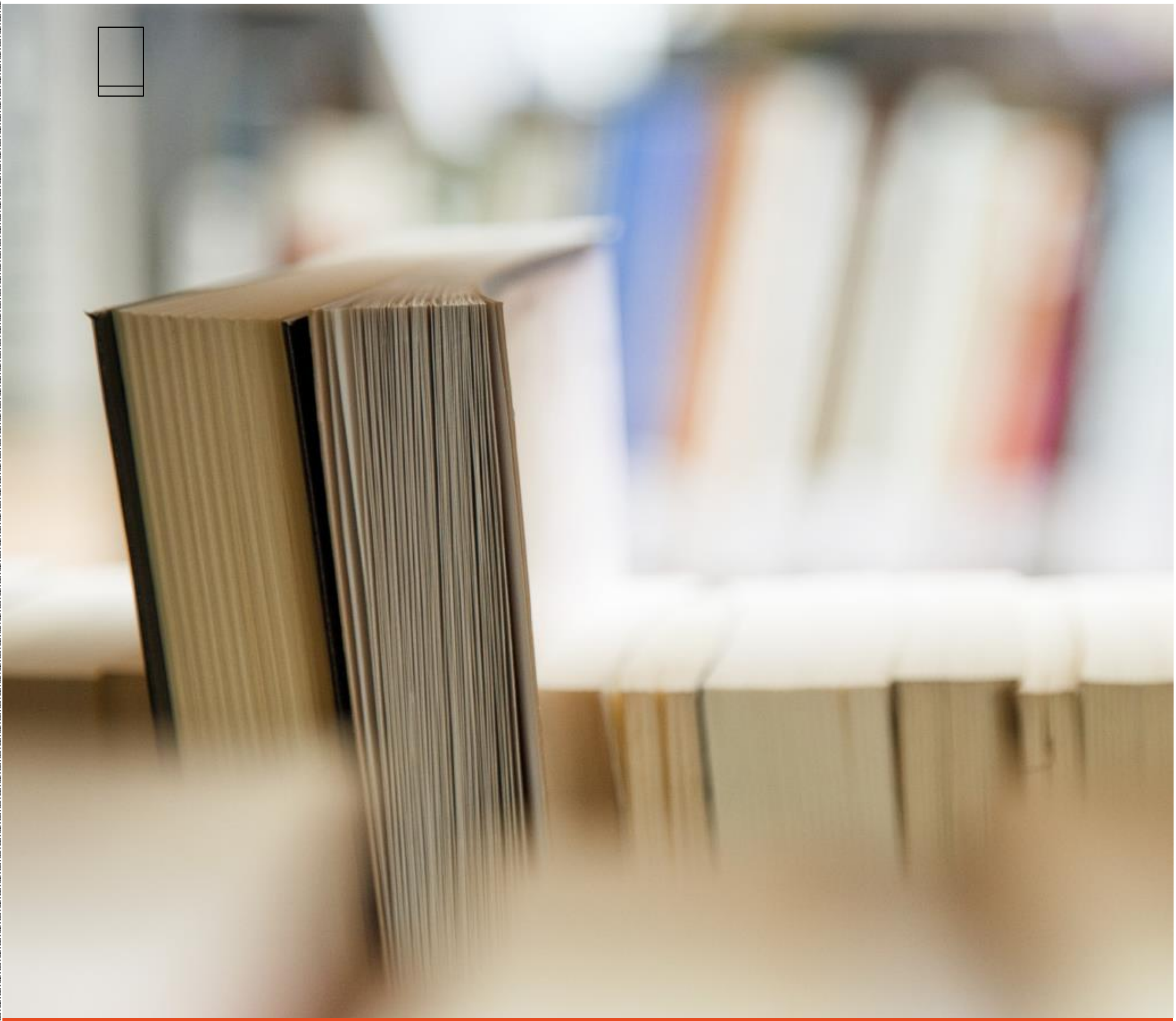

DIMENSION & MODEL ANALYSIS

DEDICATED TO ALL GATE AE ASPIRANTS

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Buckingham's Pi-Theorem

If there are n variables in a dimensionally homogeneous equation and if these variables contain m primary dimensions, then the variables can be grouped in to (n-m) non-dimensional parameters. The non-dimensional groups are called Pi-terms.

Mathematically:

$$f\left(x_1, x_2, x_3, x_4........x_n\right) = 0$$

Dimensionless Pi-terms:

$$\phi = \left[\pi_1, \pi_2, \pi_3........\pi_{n-m}\right]$$

Where m represents the fundamental dimensions such as mass, length, and time or force, length and time.

Model studies: Prototype is the full-size structure employed in the actual engineering design. The prototype operates under the actual working conditions. Model is a system by whose operation the characteristics of other similar system can ascertained.

Similitude: **Similitude refers to the theory and art of predicting prototype conditions from model observations. It prescribes the relationship between a full-scale flow and flow involving smaller but geometrically similar boundaries. The results obtained from experiments on models can be applied to the prototype only if a complete similarity exists between the model and prototype, and for that the two systems must be:**

(i) Geometrically similar

(ii) Kinematically similar

(iii) Dynamically similar

1. Geometrical Similarity:

For geometrical similarity to exist, the ratio of corresponding length dimension between the model and prototype must be same.

The geometric parameters are length, height, area, volume, diameter.

Area scale ratio:

$$\frac{l_m}{l_p} = \frac{b_m}{b_p} = \frac{h_m}{h_p} = L_r$$

Area scale ratio:

$$A_r = \frac{A_m}{A_p} = \frac{l_m \times b_m}{l_p \times b_p} = (L_r)^2$$

Volume scale ratio:

$$V_r = \frac{V_m}{V_p} = \frac{l_m \times b_m \times h_m}{l_p \times b_p \times h_p} = (L_r)^3$$

2. Kinematic Similarity:

Kinematic parameters are velocity, acceleration, discharge.

- At all corresponding points in the model and prototype the ratio of velocity as well as acceleration must be same (both in magnitude and direction).
- Such similarity can be attained if flow nets for the model and proto type are generally similar

Velocity ratio $= \frac{V_m}{V_p} = V_r = \frac{L_r}{T_r}$

Acceleration ratio $= \frac{a_m}{a_p} = a_r = \frac{L_r}{T_r^2}$

Discharge ratio $= \frac{Q_m}{Q_p} = Q_r = \frac{L_r^3}{T_r}$

- Geometric similarity is a prerequisite for kinematic similarity.

3. Dynamic Similarity:

For dynamic similarity to exist between model and prototype, identical type of forces (viscous, pressure, elastic etc.) must be parallel and must bear the same ratio at all corresponding sets of points

Dynamic parameters are force and power

Force ratio$= \frac{F_m}{F_p} = \rho_r V_r a_r = \frac{\rho_r L_r^4}{T_r^2}$

Power ratio $= \frac{P_m}{P_p} = F_r V_r = \frac{\rho_r L_r^5}{T_r^3}$

- Both geometric and kinematic similarities are prerequisite for dynamic similarity.

• For kinematic similarity:	Geometrical similarity must exist.
• For dynamic similarity:	1. Geometrical similarity must exist. 2. Kinematic similarity must exist but is not the sufficient condition for dynamic similarity.

For complete similarity, geometrical, kinematic and dynamic similarity must exist:

- A model can be larger than prototype. Normally larger models are made when
  (a) Flow field is very small
- Models in these cases are made larger and slower so that experimental measurements and flow visualization are easier.

Examples are:

- Modeling a hard disk drive.
- Modeling insect fight
- Modeling the setting of very small particle in air or water.
- Modeling the motion of water droplet in clouds.
- Modeling flow through fine tubes.
- Modeling biological system like blood through capillaries, flow in bronchi of lungs etc.

Different type of forces acting on a fluid mass

1. **Inertia Force:** It is a product of mass and acceleration. It always exists in the fluid flow problems.

$$F_i = m\frac{V}{t} = \rho L^3 \frac{V}{t} = \rho L^2 \frac{L}{t} V = \rho L^2 V^2$$

$$\boxed{F_i = \rho L^2 V^2}$$

2. **Viscous Force:**

$$F_\mu = \tau A = \mu \times \frac{V}{L} . L^2$$

$$\boxed{F_\mu = \mu V L}$$

3. **Gravity Force:**

$$F_g = mg = \rho L^3 g$$

$$\boxed{F_g = \rho L^3 g}$$

4. **Pressure Force:**

$$F_p = PA = PL^2$$

$$\boxed{F_p = PL^2}$$

5. **Surface Tension Force:**

$$\boxed{F_\sigma = \sigma L}$$

6. **Elasticity Force:**

$F_e = K.A$

$$F_e = K.L^2$$

K → Bulk modulus

IMPORTANT DIMENSION LESS NUMBER [OR NON-DIMENSIONAL PARAMETERS]

1. Reynolds Number

$$R_e = \frac{\text{Inertial force}}{\text{Viscous force}} = \frac{\rho L^2 V^2}{\mu VL} = \frac{\rho VL}{\mu}$$

$$R_e = \frac{\rho VL}{\mu}$$

2. Froude's Number

$$F_e = \sqrt{\frac{\text{Inertial force}}{\text{gravity force}}} = \sqrt{\frac{\rho L^2 V^2}{\rho L^3 g}} = \frac{V}{\sqrt{gL}}$$

$$F_e = \frac{V}{\sqrt{gL}}$$

3. Euler's Number

$$Eu = \sqrt{\frac{\text{Inertial force}}{\text{pressure force}}} = \sqrt{\frac{\rho L^2 V^2}{PL^2}}$$

$$Eu = \frac{V}{\sqrt{P/\rho}}$$

4. Weber Number

$$W_e = \frac{\text{Inertial force}}{\text{Surface tension force}} = \sqrt{\frac{\rho L^2 V^2}{\sigma L}}$$

$$\boxed{W_e = \frac{V}{\sqrt{\sigma / \rho L}}}$$

5. Mach Number

$$M = \sqrt{\frac{\text{Inertial force}}{\text{Elastic force}}} = \sqrt{\frac{\rho L^2 V^2}{KL^2}}$$

$$\boxed{M = \frac{V}{\sqrt{k / \rho}} = \frac{V}{C}}$$

C → Velocity of sound in the medium

MODEL LAWS

For dynamic similarity between model and prototype, ratio of dimension less number (Re, Fe, We, Eu, M) should be same for model and prototype.

- But it is difficult to satisfy all the force ratio. Hence models are designed on the basis of ratio of forces which are dominating in the phenomenon.
- Thus, we have various model laws depending on the forces which are dominant in the flow.

(1) Reynold's model Law

- The situation where apart from inertial force, only viscous forces are dominant, Reynold's model law is used i.e. Reynold's number of models should be equal to Reynold's number of prototypes.

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p} \rightarrow \boxed{\frac{\rho_r V_r L_r}{\mu_r} = 1}$$

Reynold's model law is used in

- Pipe flow (where viscosity has significant effect like laminar flow).
- Flow around completely submerged objects like submarines, airplanes, automobiles.
- Parachutes when there is drag (i.e. when not falling in vacuum)

Thus, in this law:

(a) $V_r = \frac{\mu_r}{\rho_r L_r}$ = Velocity ratio

(b) $T_r = \frac{L_r}{V_r} = \frac{\rho_r L_r^2}{\mu_r}$ = Time ratio

(c) $a_r = \frac{V_r}{T_r} = \frac{\mu_r^2}{\rho_r^2 L_r^3}$ = Acceleration ratio

(d) $F_r = m_r a_r = \frac{\rho_r L_r^3 \mu_r^2}{\rho_r^2 L_r^3} = \frac{\mu_r^2}{\rho_r}$ = Force ratio

(e) $P_r = F_r V_r = \frac{\mu_r^3}{\rho_r^2 L_r}$ = Power ratio

(f) $Q_r = \frac{L_r^3}{T_r} = V_r \times L_r^2 = \frac{\mu_r}{\rho_r L_r} \times L_r^2 = \frac{\mu_r L_r}{\rho_r}$ = Discharge ratio

(2) Froude's Law

- The law is applicable when in addition to internal force, gravity forces are important.

  In this case, $(F_e)_m = (F_e)_p$

$$\frac{V_r}{\sqrt{g_r L_r}} = 1$$

If model and prototype are at same location i.e. if g is constant, then

$$\frac{V_r}{\sqrt{L_r}} = 1$$

Froude's law is applicable in

(a) Free surface flow such as flow over spill way, weirs, channels (open channels) hydraulic jumps.

(b) Flow of liquid jets from orifices.

(c) Flow over weir and notches.

(d) Motion of ship in rough and turbulent sea (motion resistance is due to waves formed in sea due to gravity.

Note: In case of ship (gravity as well as viscous forces both are permanent) hence Reynold's and Froude law both should be used.

In the case

1. $V_r = \sqrt{L_r}$ = velocity ratio
2. $T_r = \frac{L_r}{V_r} = \frac{L_r}{\sqrt{L_r}} = \sqrt{L_r}$ = Time ratio
3. $a_r = \frac{V_r}{T_r} = \frac{\sqrt{L_r}}{\sqrt{L_r}} = 1$ = Acceleration ratio
4. $F_r = m_r a_r = \rho_r L_r^3 \times 1 = \rho_r L_r^3$ = Force ratio
5. $P_r = F_r V_r = \rho_r \times L_r^3 \times \sqrt{L_r} = \rho_r L_r^{7/2}$ = power ratio
6. $Q_r = \frac{L_r^3}{T_r} = \frac{L_r^3}{\sqrt{L_r}} = L_r^{2.5}$ = discharge ratio

(3) Euler's model law

- When apart from inertial forces, only pressure force is dominant, this law is applicable.
- In this case, $(E_u)_m = (E_u)_p$

$$\frac{V_r}{\sqrt{P_r / \rho_r}} = 1$$

This law is applicable

- In pipe flow at high pressure (in which viscous effect is negligible as in turbulent flow)
- In case of cavitation.
- Pressure due to sudden closure of valve.

(4) Weber model Law

When apart from inertial forces, surface tension force is dominant.

$$(W_e)_m = (W_e)_p$$

$$\frac{V_r}{\sqrt{\frac{\sigma_r}{\rho_r L_r}}} = 1$$

It is applicable in

(a) Capillary rise in narrow passages.

(b) Flow over weirs for small head.

(c) Rising bubble, seepage flow through soil.

(d) Flow of blood in veins and arteries.

(5) Mach Law

When apart from inertial force, compressibility forces are dominant, we use Mach law.

- Compressibility forces are predominant when **Mach no** $\geq$ **0.3**

  i.e. speed of movement $\geq$ (0.3 $\times$ velocity of sound)

  Mach no < 1 is called Subsonic

  Mach no > 1 is called Supersonic

  Mach no = 1 is called sonic

  Mach no >> 1 is called Hypersonic

  Mach no < 0.3 → Compressibility effect can be neglected

In this case,

$$(M)_m = (M)_p$$

$$\frac{V_r}{\sqrt{\frac{K_r}{\rho_r}}} = 1$$

Applicable in

- *Aerodynamic testing such as launching of missiles, rockets.*
- *Under water testing of torpedoes.*
- *Water hammer pressure.*
- *Flow of gases through pipe at high velocity.*