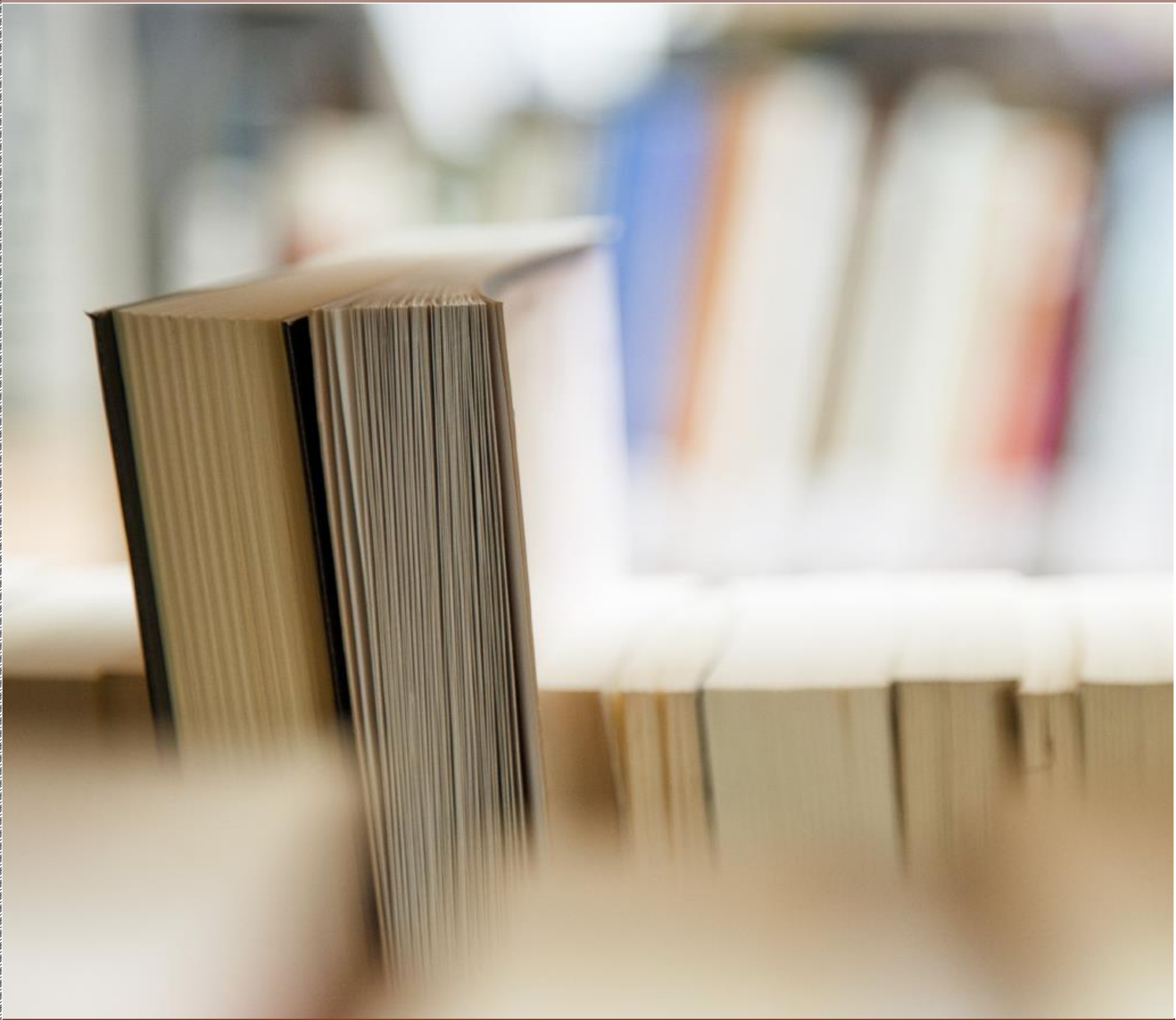
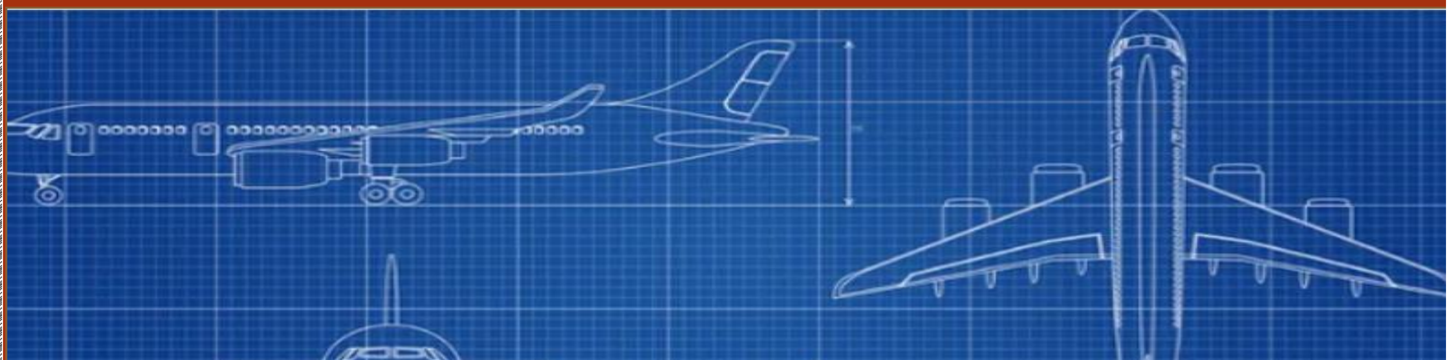


ROCKET PROPULSION

DEDICATED TO ALL GATE AE ASPIRANTS



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// REFERENCES //

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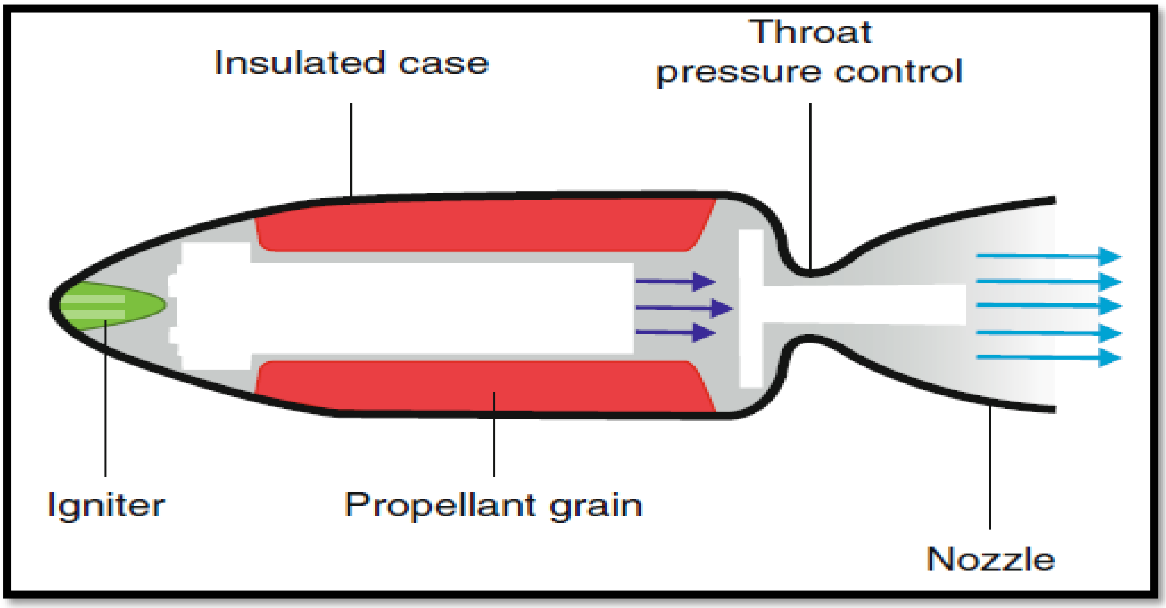
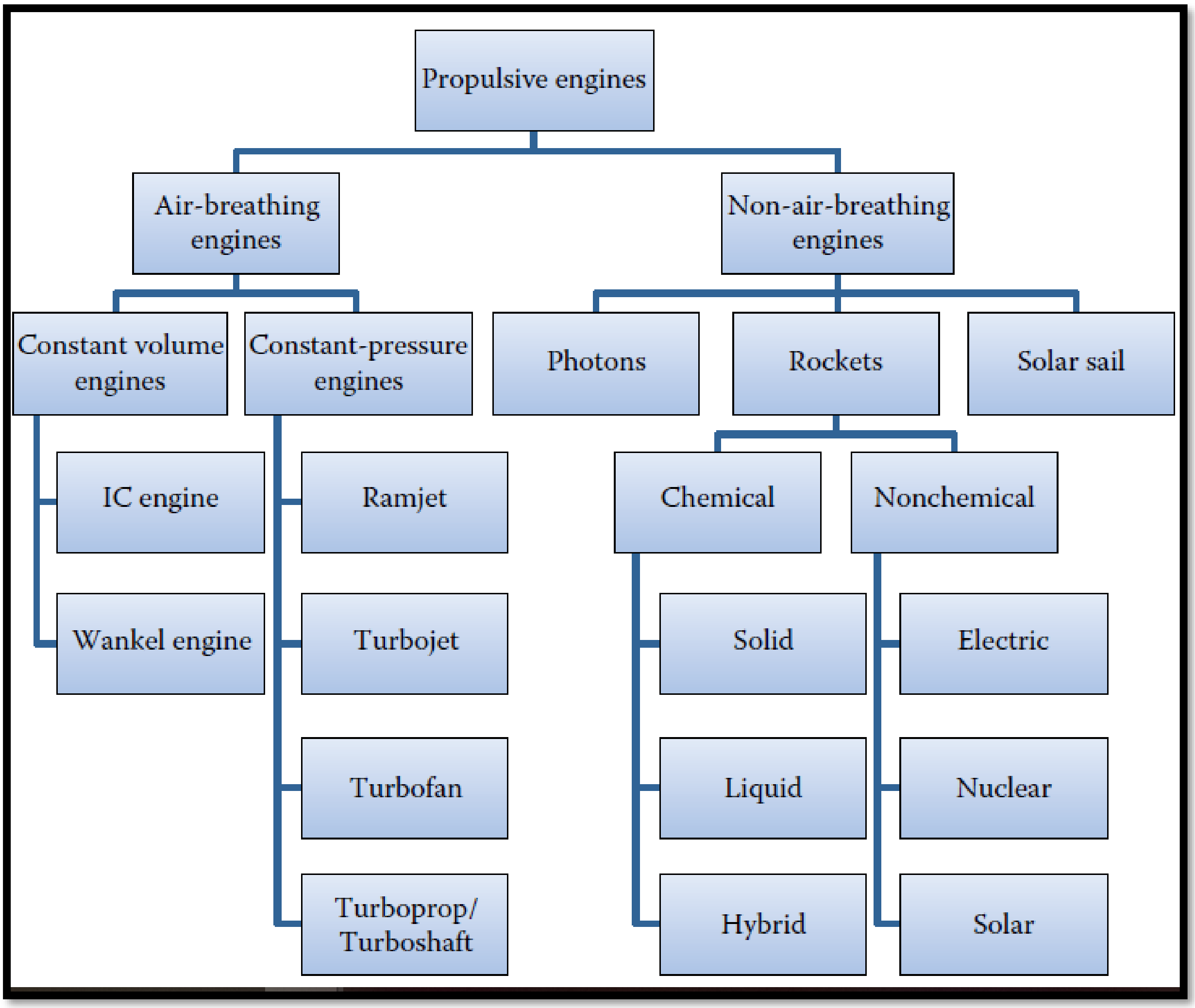
ROCKET PROPULSION

A rocket is a propulsive device that produces a thrust force F on a vehicle by ejecting mass a high relative velocity.



Classification of propulsive devices:

The propulsive devices can be broadly divided into two categories: air-breathing and non-air breathing engines



Comparison of Air-Breathing and Rocket Engines

Air-Breathing Engine

1. It uses atmospheric air for combustion.
2. It cannot operate in space (vacuum).
3. Its performance is dependent on altitude and flight speed.
4. The thrust developed by this engine is dependent on flight speed.
5. It cannot operate beyond supersonic speed ($M = 5.0$).
6. Rate of climb decreases with altitude.
7. Flight speed is always less than jet velocity.

Rocket Engine

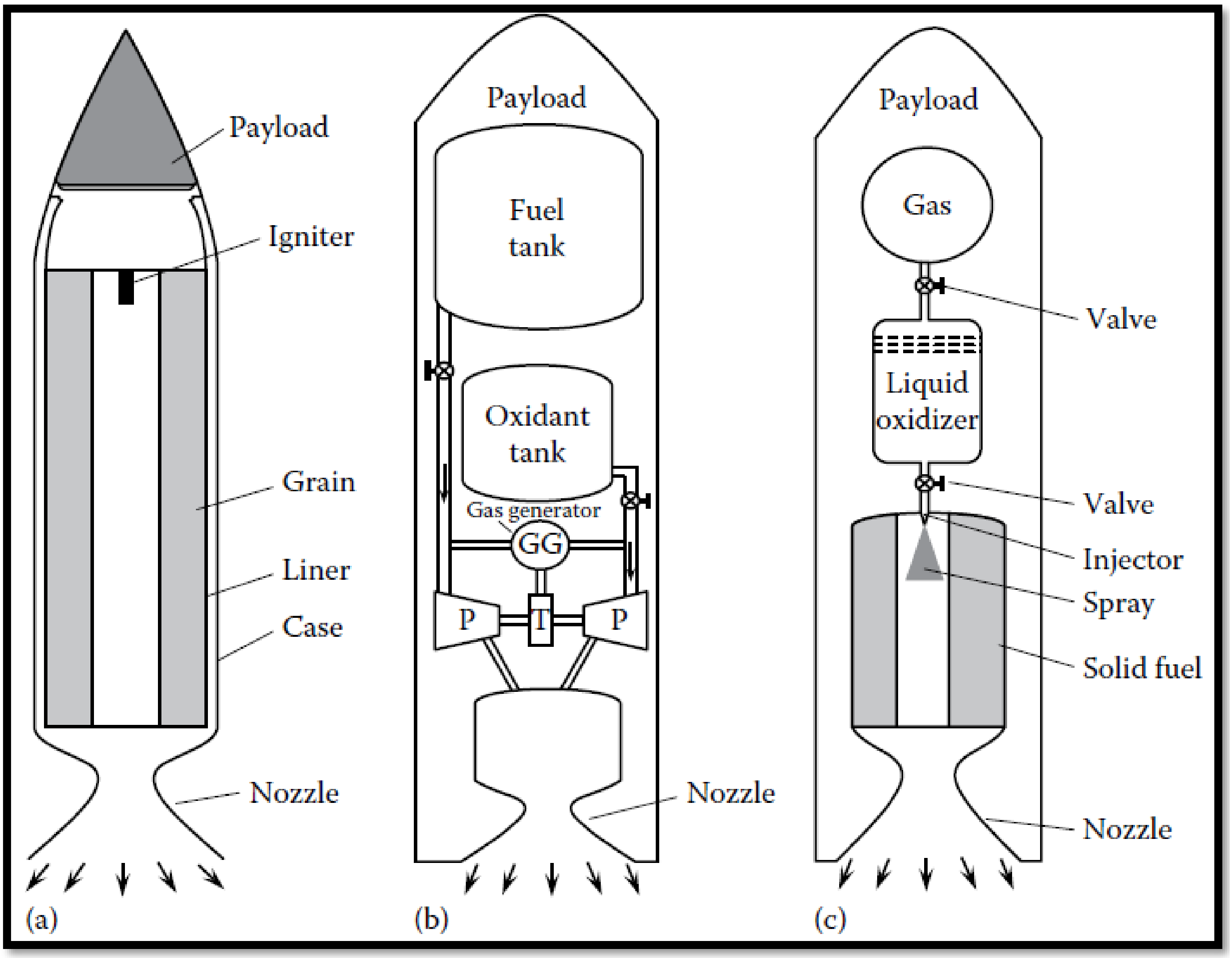
1. It carries its own oxidizer and fuel
2. It can operate in atmosphere and space
3. Its performance does not depend on altitude and flight speed
4. The thrust of this engine is independent of flight speed
5. It can operate at any flight velocity (Mach number: M = 0–25).
6. Rate of climb increases with altitude

Types of chemical Rocket Engines

(1) Solid propellant

(2) Liquid propellant

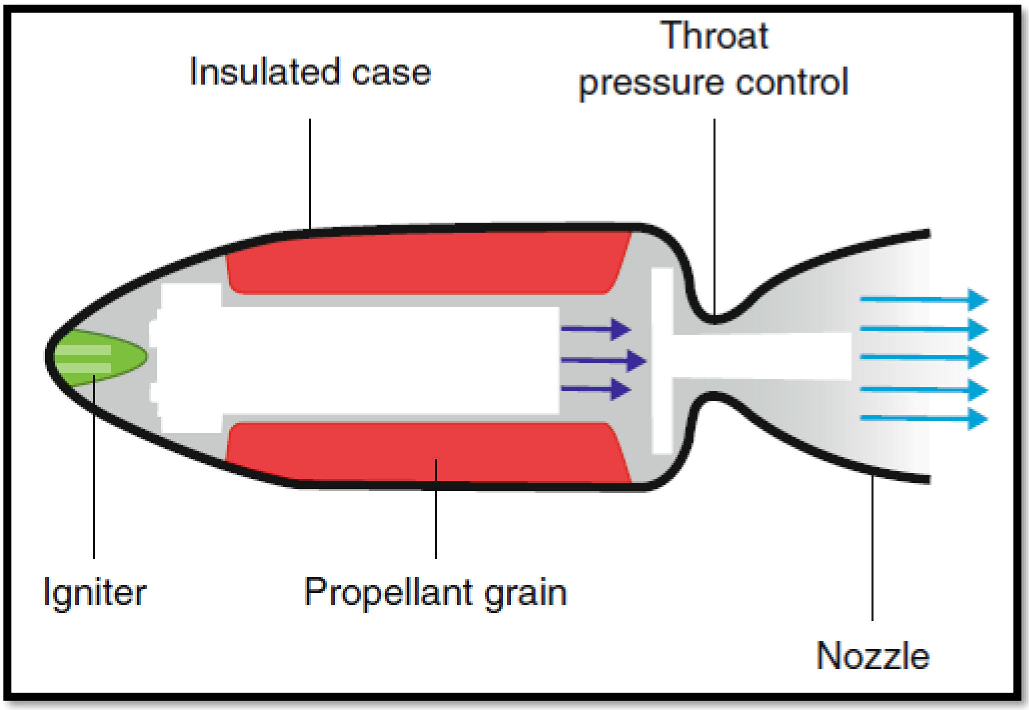
(3) Hybrid propellant.



(a) Solid propellant (b) Liquid propellant (c) Hybrid propellant.

Solid-Propellant Rocket Engines

- When solid propellants are used, both fuel and oxidizer are contained in the same casing or combustion chamber.
- The propellant charge is called the "grain" and it contains the entire chemical ingredients for complete burning.
- Once ignited, it usually burns smoothly at a nearly constant rate on the exposed surface of the charge. Because there are no feed systems or valves such as there are in liquid units, solid-propellant rockets are relatively simple in construction.
- SPRE is particularly well suited for developing very high thrust within a short interval of time, particularly in the booster phase.
- With recent advancements in propellant chemistry, it can be used for fairly long burning time (sustainers).



Advantages:

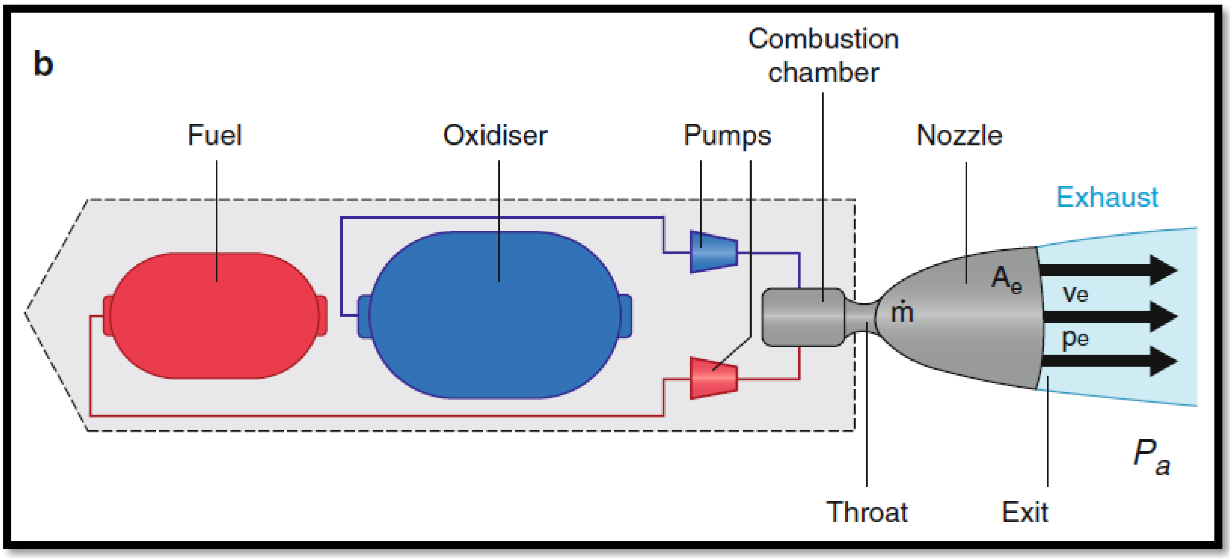
1. It is simple to design and develop.
2. It is easier to handle and store unlike liquid propellant.
3. Detonation hazards of many modern SPREs are negligible.
4. Better reliability than Liquid Propellant Rocket Engine (LPRE) (>99%).
5. Much easier to achieve multi-staging of several motors.
6. The combination pressure in SPREs is generally higher than in LPREs since it is not subject to the limitation of a feed system.
7. Development and production cost of SPREs is much smaller than that of LPREs, especially in the high-thrust bracket.

Disadvantages:

1. It has lower specific impulse compared to LPREs and hybrid propellant rocket engines (HPREs).
2. It is difficult to turn off its operation unlike in an LPRE.
3. Transport and handling of solid propellants are quite cumbersome.
4. It is difficult to use the thrust vector control and thrust modulation.
5. The cracks on the propellant can cause an explosion.
6. Careful design of the nozzle is required as active cooling cannot be used.
7. The erosion of the throat area of the nozzle due to high-temperature solid particles can affect its performance adversely

Liquid Propellant Rocket Engines

- LPREs are stored in separate tanks unlike SPRE, one can achieve a higher level of thrust and is thus considered to be more powerful than an SPRE.
- It is preferred for large spacecraft and ballistic missiles. However, the design of an LPRE is quite complex and requires specialized nozzles. Compared to other types of chemical rocket engines, LPREs are compact, light, economical, and highly reliable. Hence, they have a wider range of both civilian and military applications.



Advantages:

1. An LPRE can be reused.
2. It provides greater control over thrust.
3. It can have higher values of specific impulse.
4. In case of emergency its operation can be terminated very easily.
5. It can be used on pulse mode.
6. It can be used for long-duration applications.
7. It is easy to control this engine as one can vary the propellant flow rate easily.
8. The heat loss from the combustion gas can be utilized for heating the incoming propellant.

Disadvantages:

1. This engine is quite complex compared to the SPRE.
2. It is less reliable as there is a possibility of malfunctioning of the turbo pump injectors and valves.
3. Certain liquid propellants require additional safety precaution.

4. It takes much longer to design and develop.
5. It becomes heavy, particularly for short-range application

Hybrid Propellant Rocket Engines:

In order to achieve better performance, elements from SPREs and LPREs are combined to devise a new engine known as the HPRE. Note that this engine can use both solid and liquid types of propellants.

It has similar features to an LPRE, namely, compact, light, economical, and highly reliable. Besides, these engines may have better performance compared to both solid and liquid engines. Hence, these engines may find a wider range of both civilian and military applications.

Advantages:

1. An HPRE can be reused.
2. It provides greater control over thrust.
3. It has relatively lower system cost compared to the LPRE.
4. It can have higher values of average specific impulse compared to the SPRE.
5. It has higher density of specific impulse than that of the LPRE. It has higher volume utilization compared to the LPRE.
6. It has start–stop–restart capability.

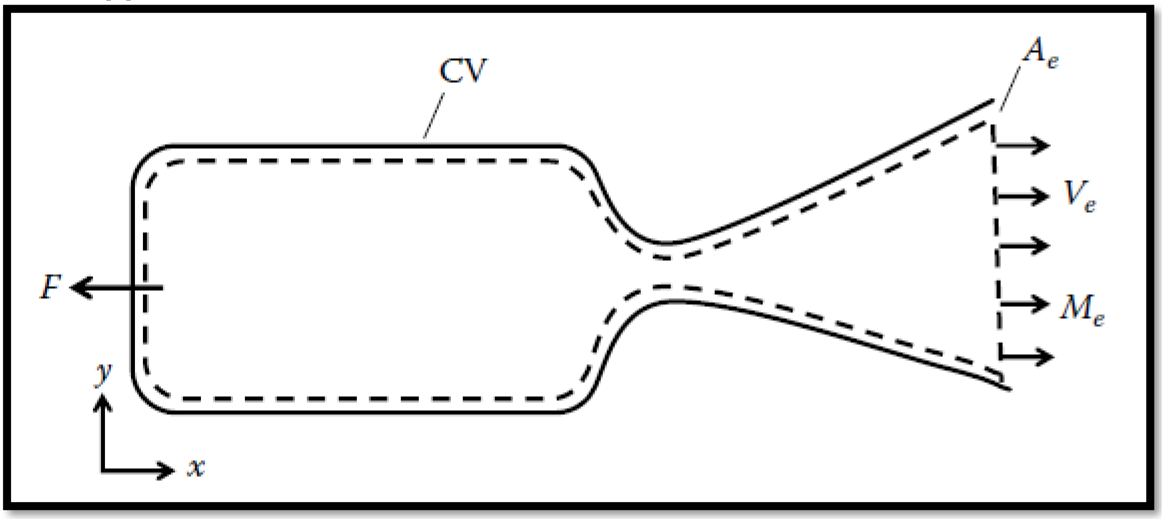
Disadvantages:

1. This engine is quite complex compared to the LPRE.
2. Its mixture ratio varies to some extent and hence it is quite difficult to achieve steady-state operation.
3. It has lower density of specific impulse compared to SPRE.
4. There is underutilization of solid fuel due to a larger sliver of residual grain at the end of the operation.
5. Certain liquid propellants require additional safety precaution.
6. It takes much longer to design and develop.
7. It becomes heavy, particularly for short-range application.

8. It has an unproven propulsion system for large-scale applications

Thrust equation for rocket engine

An expression for the thrust developed by a rocket engine under static condition can be obtained by applying the momentum equation. For this, let us consider a control volume (CV), as shown in following figure The propulsive thrust "*F*" acts in a direction opposite to V_e The reaction to the thrust "*F*" on the CV is opposite to it.



➢ The momentum equation for such CV is given by

$$\frac{d}{dt}\int_{cv}\rho V_x dV+\int_{cs}V_x\left(\rho V_x\cdot n\right)dA=\sum F_x \text{ ....(1)}$$

- As the flow is steady in nature, we can neglect the unsteady term i.e.

$$\frac{d}{dt}\int_{cv}\rho V_x dV=0$$

- Let us now evaluate the momentum flux term and sum of forces acting on CV

$$\int_{cs}V_x\left(\rho V_x\cdot n\right)dA=\int V_x d\dot{m}=\dot{m}V_e \text{ ....(2)}$$

$$\sum F_x=F+p_aA_e-p_eA_e \text{ ....(3)}$$

By solving equation (1)(2) and(3)

$$F=\dot{m}V_e-p_aA_e+p_eA_e=\dot{m}V_e+A_e\left(p_e-p_a\right)$$

Or

$$F=\dot{m}V_e+A_e\left(p_e-p_a\right)$$

Where
F is the thrust
V_e is the velocity component at exit of nozzle

$\dot{m}$ is the mass flow rate of propellant
p_e is the pressure at exit plane of nozzle
p_a is the ambient pressure

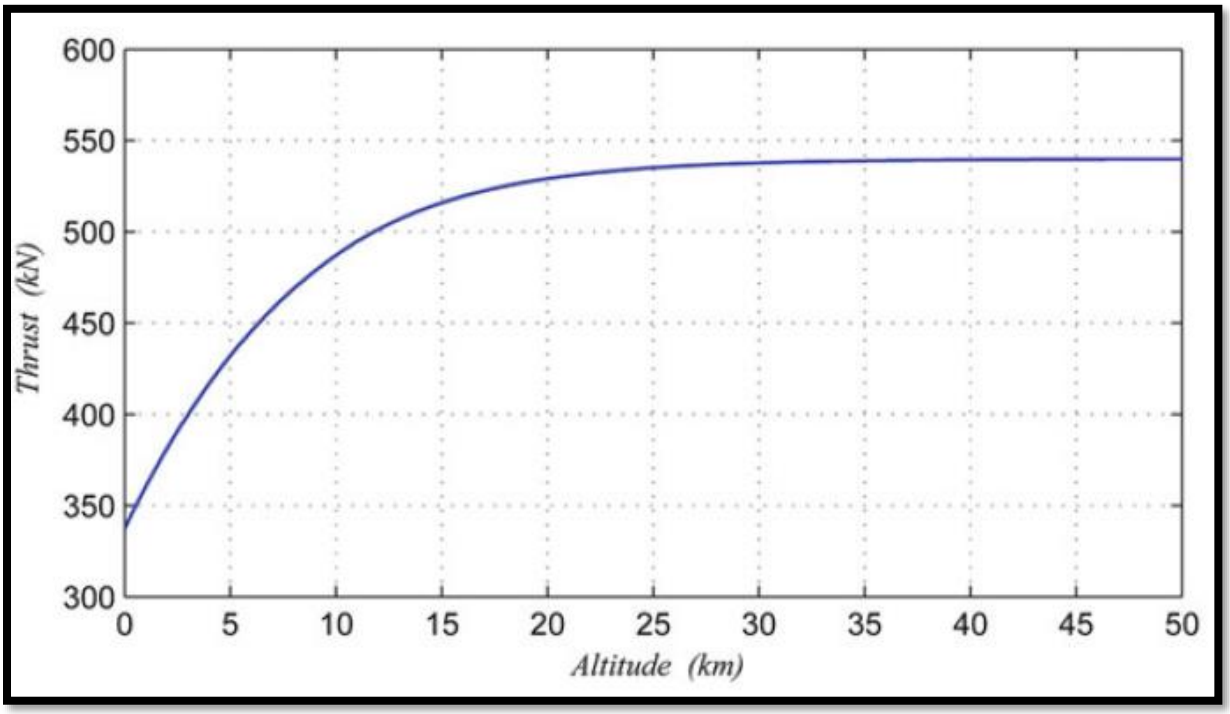
$$F = \dot{m}_p V_e + A_e\left(p_e - p_a\right) \quad \text{...........(4)}$$

Where the first and second terms represent the momentum contribution and pressure components of thrust, respectively. The thrust force is independent of vehicle speed. It depends on the atmospheric pressure p_a as well as the exhaust parameters: mass flow rate $\dot{m}$ speed V_e and pressure p_e

In the vacuum of space $p_a = 0$ the thrust is expressed as:

$$F = \dot{m}V_e + p_e A_e$$

For a given design of nozzle, the value of exit pressure p_e is fixed. Thus, during flight through the atmosphere, the ambient pressure p_a decreases continuously. As a result, the thrust increases with altitude. However, beyond a certain altitude, the variation is negligible and the thrust is nearly constant



Maximum Thrust

We need to evaluate the condition under which maximum thrust can be achieved for a given chamber pressure and mass flow rate with a fixed throat area of nozzle. We can obtain this condition by differentiating the thrust expression, **Equation (4)**, given in the following:

$$dF = \dot{m} dV_e + dA_e \left(p_e - p_a \right) + A_e dp_e \quad \text{.........(5)}$$

Note that mass flow rate remains constant $\dot{m}$ By invoking momentum equation for one dimensional steady inviscid flow, we can have

$\dot{m} dV_e = -A_e dp_e$ .... now putting this value in equation (5)

$dF = dA_e \left(p_e - p_a \right)$ for maximum thrust derivative must be zero.

Hence

$$\frac{dF}{dA_e} = p_e - p_a = 0$$

$$p_e = p_a$$

The condition $p_e = p_a$ is called optimum expansion because it corresponds to maximum thrust for the given chamber conditions. The maximum thrust can be obtained only when the nozzle exhaust pressure is equal to the ambient pressure.

Now expression for maximum thrust will be:

$$F_{\max} = \dot{m} V_e$$

This condition provides a maximum thrust for a given chamber pressure and propellant mass flow rate. The rocket nozzle in which this condition is achieved is known as optimum expansion ratio nozzle.

Effective Exhaust Velocity

The effective exhaust velocity V_{eq} is often used to compare the effectiveness of different
Propellants in producing thrust in rocket engine.
We know that velocity profile at the exit of nozzle need not be one dimensional in nature in the practical situation. It is quite cumbersome to determine the velocity profile at the exit of a rocket nozzle. In order to tackle this problem, we can define effective exhaust velocity by using equation (4)

$$V_{eq} = \frac{F}{\dot{m}} = V_e + \frac{A_e\left(p_e - p_a\right)}{\dot{m}}$$

$$V_{eq} = V_e + \frac{A_e\left(p_e - p_a\right)}{\dot{m}}$$

where V_{eq} is the effective exhaust velocity that would be produced equivalent thrust which could have produced due to both momentum and pressure components of thrust.

Note: When the exit pressure is same as that of the ambient pressure, the effective exhaust velocity V_{eq} becomes equal to nozzle exit velocity V_e Otherwise, the effective exhaust velocity V_{eq} is less than the nozzle exit velocity V_e.

Effective exhaust velocity in terms of exit mach number:

$$V_{eq} = V_e + \frac{A_e\left(p_e - p_a\right)}{\dot{m}}$$

The mass flow rate is expressed by $\dot{m} = \rho_e V_e A_e$

$$V_{eff} = V_e\left[1 + \frac{\left(p_e - p_a\right)A_e}{V_e\left(\rho_e V_e A_e\right)}\right]$$

$$V_{eff} = V_e\left[1 + \frac{\left(p_e - p_a\right)}{V_e\left(\rho_e V_e\right)}\right]$$

$$V_{eff} = V_e\left[1 + \frac{p_e}{\rho_e V_e^2}\left(1 - \frac{p_a}{p_e}\right)\right]$$

$$V_{eff} = V_e\left[1 + \frac{RT_e}{\gamma RT_e M_e^2}\left(1 - \frac{p_a}{p_e}\right)\right]$$

$$V_{eff} = V_e\left[1 + \frac{1}{\gamma M_e^2}\left(1 - \frac{p_a}{p_e}\right)\right]$$

Hence

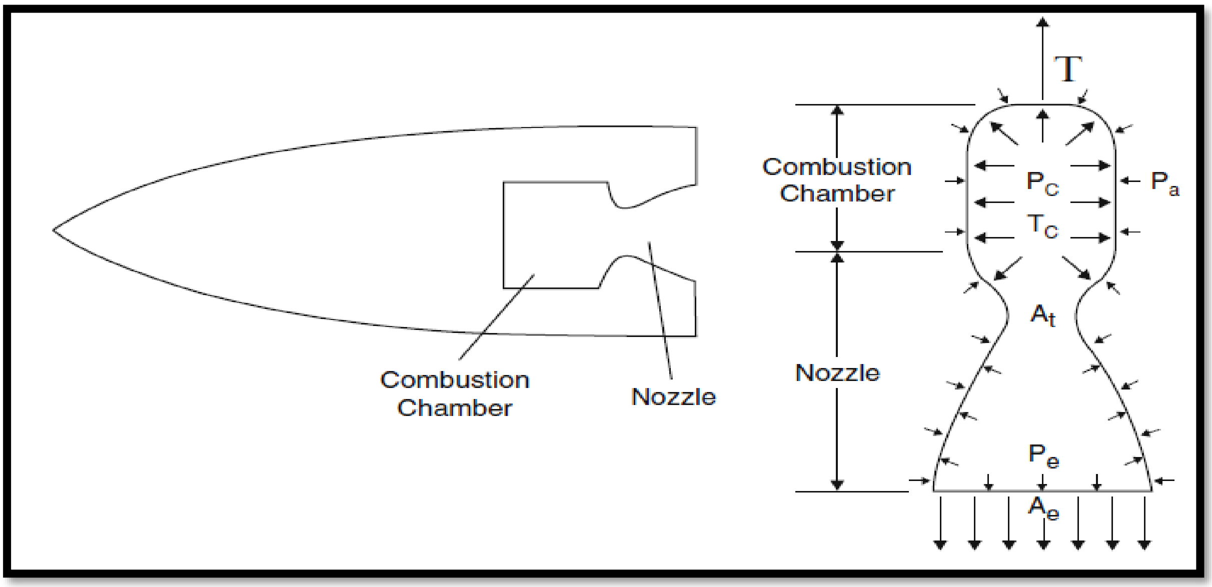
$V_{eff} = V_e$ *When* $p_e = p_a$

$$V_{eff} = V_e\left[1 + \frac{1}{\gamma M_e^2}\left(1 - \frac{p_a}{p_e}\right)\right]$$

Exhaust Velocity:

The propellants enter the combustion chamber, mix, and are ignited. The gas produced is heated by chemical energy released during combustion and expands through nozzle. Applying the first law of thermodynamics, the change in enthalpy of exhaust gases is equal to its kinetic energy. The temperature of gases in combustion chamber T_c will be equal to the total temperature of gases exhausting the nozzle $T_{0e} = T_c$

The gain in kinetic energy of exhaust gases per unit mass =The change in specific enthalpy



$$\frac{1}{2}V_e^2 = c_p \Delta T = c_p \left(T_{0e} - T_e\right)$$

$$V_e = \sqrt{2c_p \left(T_{0e} - T_e\right)}$$

$$V_e = \sqrt{2c_p T_{0e}\left(1 - \frac{T_e}{T_{0e}}\right)}$$

$$V_e = \sqrt{2\frac{\gamma R}{\gamma - 1} T_{0e}\left[1 - \left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$V_e = \sqrt{2\frac{\gamma}{\gamma - 1}\left(\frac{R_u}{M}\right) T_{0e}\left[1 - \left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

Exhaust velocity depends on following parameters:

1. Molecular weight of the exhaust gases:
2. Combustion chamber pressure
3. Combustion chamber temperature
4. Exit pressure of nozzle
5. Specific heat ratio of exhaust gases

Maximum exhaust speed

The maximum exhaust speed occurs when $p_e = 0$

$$V_{e,MAX} = \sqrt{2\frac{\gamma}{\gamma-1}\left(\frac{R_u}{M}\right)T_{0e}}$$

$$V_{e,MAX} = \sqrt{2\frac{\gamma}{\gamma-1}\left(\frac{R_u}{M}\right)T_c}$$

Ratio of Maximum exhaust speed to Exhaust Velocity:

$$\frac{V_{e,MAX}}{V_e} = 1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}$$

Problem:

A booster rocket engine with nozzle exit diameter of 225 mm is designed to propel a satellite to altitude of 20 km. The chamber pressure at 12 MPa is expanded to exit pressure and temperature of 105 kPa and 1400 K, respectively. If the mass flow rate happens to be 15 kg/s, determine the exit jet velocity, effective jet velocity, and thrust at Altitude = 20 km. MW = 25 kg/kmol.

Solution:

At 20 km altitude, $P_a = 5.53$ kPa.

Assuming one-dimensional flow at the exit of nozzle, and assuming ideal gas law, we can evaluate exhaust velocity from mass flow rate at exit as

$$V_e = \frac{\dot{m}}{\rho_e A_e} = \frac{4 R_e T_e \dot{m}}{MW P_e \pi D_e^2} = \frac{4 \times 8.314 \times 1400 \times 15}{25 \times 105 \times 3.14 \times (0.225)^2} = 1673.65 \text{ m/s}$$

$$V_{eq} = V_e + \frac{(P_e - P_a)}{\dot{m}} A_e$$
$$= 1673.65 + \frac{(105 - 5.53) \times 10^3}{15} \frac{3.14}{4} (0.225)^2$$
$$= 1673.65 + 263.5 = 1937.15 \text{ m/s}$$

Note that the equivalent velocity is almost the same as the exit velocity, indicating that thrust due to momentum will be predominant. We can evaluate total thrust as

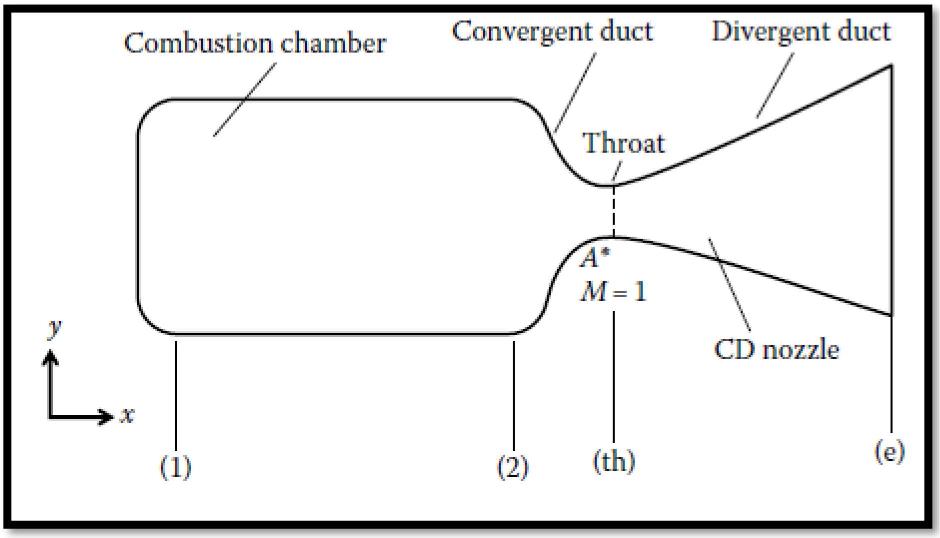
$$F = \dot{m} V_{eq} = 29057.25 = 29.06 \text{ kN}.$$

Mass Flow Rate through the C-D Nozzle

The mass flow rate through the CD nozzle under steady-state condition is dependent on the pressure, temperature of combustion chamber, and cross-sectional area. By considering 1-D and steady isentropic flow, we can invoke the continuity equation as follows

$$\dot{m} = \rho A V$$

Where



ρ is the density
A is the cross-sectional area
V is the flow velocity at any location in the nozzle

As the flow is isentropic in nature, we can use the isentropic relation for density in terms of pressure as follows

$$\frac{\rho_e}{\rho_{0e}} = \left(\frac{p_e}{p_{0e}}\right)^{1/\gamma}$$

$$\rho_e = \rho_{0e}\left(\frac{p_e}{p_{0e}}\right)^{1/\gamma}$$

By the mass flow expression putting the value of exit velocity and density in following expression

$$\dot{m} = \rho_e A_e V_e$$

$$\dot{m} = \rho_e A_e \sqrt{2\frac{\gamma}{\gamma-1}\left(\frac{R_u}{M}\right)T_{0e}\left[1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

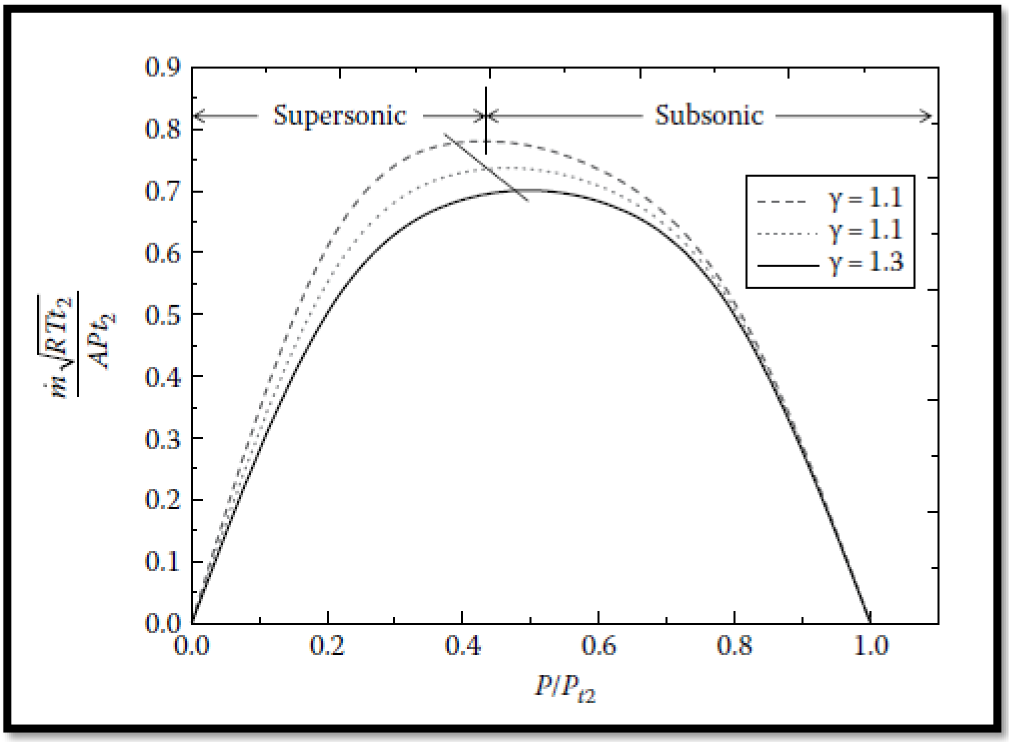
$$\frac{\dot{m}}{A_e} = \frac{p_{0e}}{RT_{0e}}\left(\frac{p_e}{p_{0e}}\right)^{\frac{1}{\gamma}} \sqrt{2\frac{\gamma}{\gamma-1}RT_{0e}\left[1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$\frac{\dot{m}}{A_e} = p_{0e}\sqrt{\frac{2\gamma}{\gamma-1}\left(\frac{1}{RT_{0e}}\right)\left(\frac{p_e}{p_{0e}}\right)^{\frac{2}{\gamma}}\left[1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

The mass flux through the CD nozzle will be dependent on the combustion chamber conditions, pressure ratio $\frac{p_e}{p_{0e}}$, and specific heat ratio γ.

Non dimensional mass flow rate:

$$\frac{\dot{m}}{A_e}\frac{\sqrt{T_{0e}}}{p_{0e}}\sqrt{\frac{R}{\gamma}} = \sqrt{\frac{2}{\gamma-1}\left(\frac{p_e}{p_{0e}}\right)^{\frac{2}{\gamma}}\left[1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$



- The variation in the non-dimensional mass flow rate through the nozzle is plot It may be observed that mass flux increases from zero value at pressure ratio p_e/p_{0e} to a maximum value at critical pressure ratio and subsequently decreases to zero value again when the pressure ratio p_e/p_{0e} becomes unity.
- As there is no pressure gradient across the nozzle at unity pressure ratio p_e/p_{0e}, no flow occurs in the nozzle, thus making the mass flux zero.
- However, when the pressure ratio p_e/p_{0e} is zero, mass flux can be zero due to infinite area, as gas has to be expanded to vacuum pressure.

- **The maximum mass flux is attained at minimum cross-sectional area of the nozzle**

Maximum mass flow rate

The pressure ratio corresponding to maximum mass flux can be obtained by differentiating the following equation wrt pressure ratio.

$$\frac{\dot{m}}{A_e}\frac{\sqrt{T_{0e}}}{p_{0e}}\sqrt{\frac{R}{\gamma}} = \sqrt{\frac{2}{\gamma-1}\left(\frac{p_e}{p_{0e}}\right)^{\frac{2}{\gamma}}\left[1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$$\frac{d(\dot{m})}{d\left(p_e/p_{0e}\right)} = 0$$ **(Condition for maximum mass flow rate)**

$$\frac{p_e}{p_{0e}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma+1}}$$

- **This relation corresponds to the critical condition under which flow is considered to be aerodynamically choked**

Calculation of maximum mass flow rate through nozzle

$$\frac{\dot{m}_{max}}{A_{th}}\frac{\sqrt{T_{0e}}}{p_{0e}}\sqrt{\frac{R}{\gamma}} = \sqrt{\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Where	$A_{th} = A^*$ $p_{0e} = p_{02}, T_{0e} = T_{02}$

- It may be noted that critical mass flux is dependent on A_{th} chamber pressure, temperature, and specific heat ratio.
- Generally, in such cases, the rocket engine nozzle is choked for the major portion of its operation.

Mass flow rate of exhaust gases from nozzle:

$$\dot{m} = \rho_e V_e A_e$$

$$\dot{m} = \rho_e A_e \sqrt{2\frac{\gamma}{\gamma-1}\left(\frac{R_u}{M}\right)T_{0e}\left[1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

Mass flow rate depends on following parameters:

1. **Molecular weight of the exhaust gases:**
2. **Combustion chamber pressure**
3. **Combustion chamber temperature**
4. **Exit pressure of nozzle**
5. **Specific heat ratio of exhaust gases**
6. **Density of exhaust gases**
7. **Exit area of the nozzle**

Rocket Performance Parameters

Some of the important performance parameters are:

1. **Total impulse**
2. **Specific impulse**
3. **Propellant consumption,**
4. **Thrust coefficient**
5. **Characteristic velocity.**

Total impulse

Impulse is imparted due to this change in momentum over a certain period of time In other words, total impulse *I*, imparted to the vehicle during its acceleration can be obtained by integrating over the burning time t_b

$$I = \int F dt = \int_0^{t_b} \dot{m}_p V_{eq} dt = m_p V_{eq}$$

$$I = m_p V_{eq}$$

Specific impulse

The specific impulse is an important performance variable. The specific impulse I_{sp} can be defined as the impulse per unit weight of the propellant

$$I_{sp} = \frac{I}{m_p g}$$

Where

I = Total impulse imparted to the vehicle during acceleration
m_p = The total mass of expelled propellant

g **=Acceleration due to gravity at earth's surface**

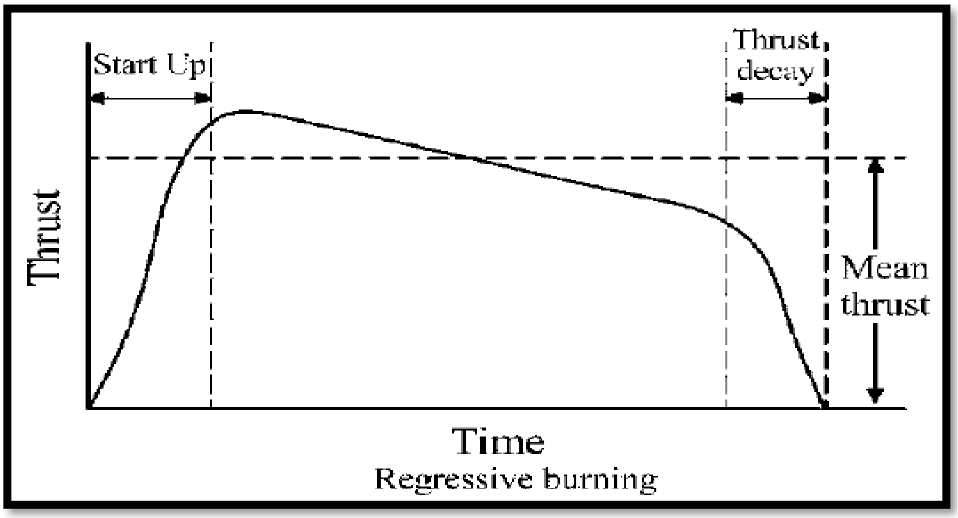
Note: Specific impulse I_{sp} represents the time average specific impulse over entire period of operation of the rocket engine. The value of specific impulse I_{sp} does vary during transient operations, namely, start, shutdown period.

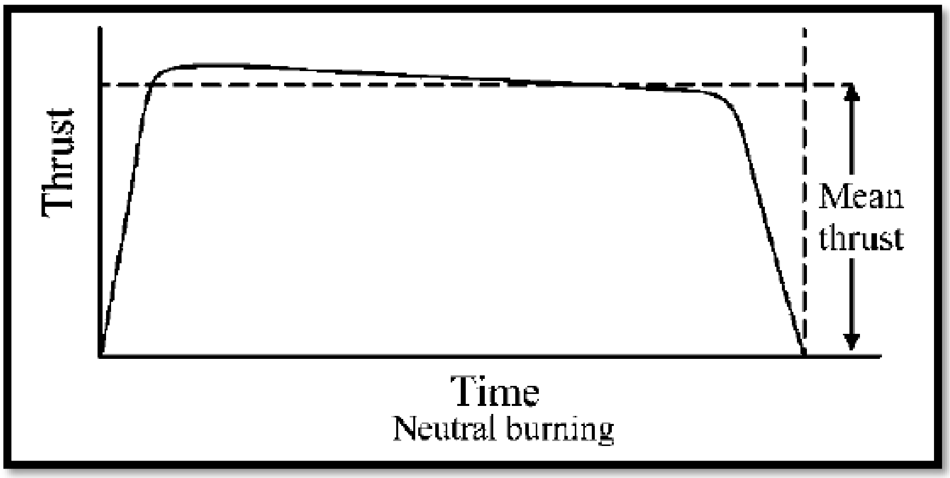
$$I_{sp} = \frac{I}{m_p g} = \frac{m_p V_{eq}}{m_p g} = \frac{V_{eq}}{g} = \frac{F}{\dot{m}_p g} = \frac{F}{\dot{w}_p}$$

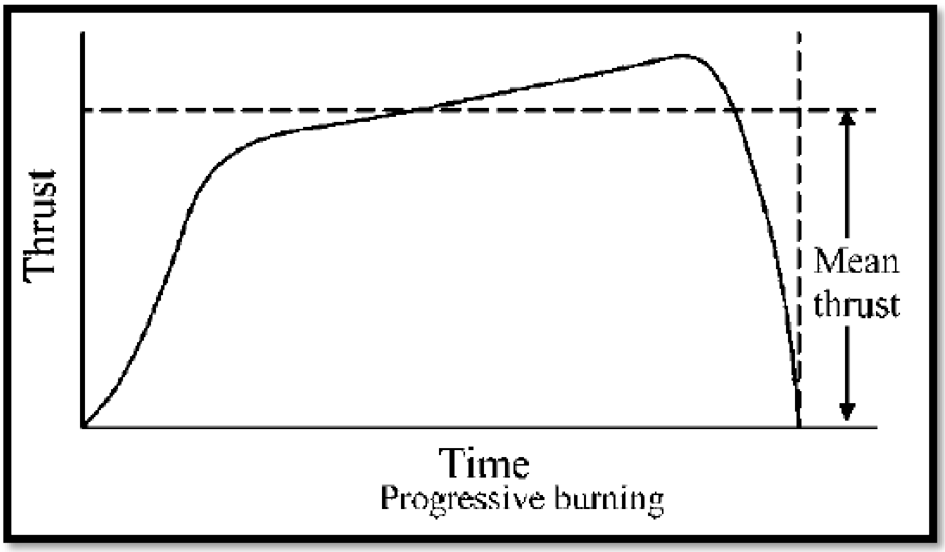
- I_{sp} can also be defined as thrust per unit weight flow rate consumption of propellant.
- It must be noted that I_{sp} does not depend on the flight velocity

Rocket Engines	I_{sp}
Solid	200–310
Liquid	300–460
Hybrid	300–500
Solar	400–700
Nuclear	600–1000
Electrical (arc heating)	400–2000

Typical thrust– time curves







Volumetric Specific Impulse

It is defined as the total impulse per unit propellant volume

$$I_V = \frac{Ft}{V_p} = \frac{m_p g I_{sp}}{V_p} = \rho_p g I_{sp}$$

Where
V_p =The propellant volume
ρ_p =Average density of propellant

- In order to ascertain the effect of engine size on the performance of rocket engine, it is important to consider volumetric specific impulse I_V in place of specific impulse.
- It is desirable to have higher volumetric specific impulse as volume required for propellant storage in the spacecraft will be smaller.

- It is essential to have higher propellant density to have higher volumetric specific impulse for a constant specific impulse system.
- That is the reason why solid-propellant rocket engine is preferred over liquid-propellant engine even though it has lower specific impulse.

Mass Flow Coefficient

It is defined as the ratio between mass flow rate of propellant m_p, and product of chamber pressure *Pc* and throat area A_t.

$$C_{\dot{m}} = \frac{\dot{m}_p}{p_c A_t}$$

It is important to have a parameter that can represent all these variables.

Thrust Coefficient

Thrust coefficient C_F which is defined as the ratio of thrust **F,** and product of chamber pressure p_c and throat area A_t of the nozzle.

$$C_F = \frac{F}{p_c\left(A_t = A^*\right)}$$

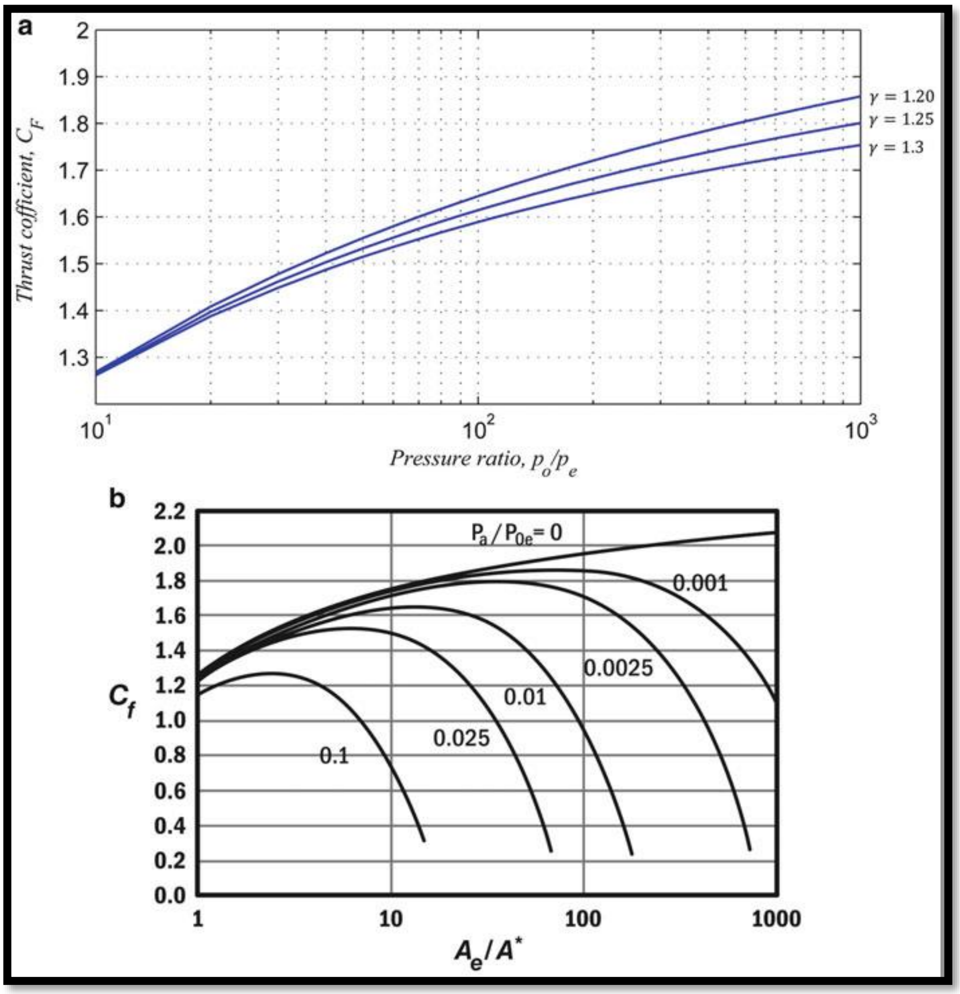
$$C_F = \sqrt{\left(\frac{2\gamma^2}{\gamma-1}\right)\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_e}{p_{0e}}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(\frac{p_e - p_a}{p_{0e}}\right)\frac{A_e}{A^*}$$

Where $p_c = p_{0e}$

Let us relate specific impulse I_{sp} to the thrust coefficient C_F and mass flow coefficient $C_{\dot{m}}$ by using specific impulse.

$$I_{sp} = \frac{F}{\dot{m}_p g} = \frac{C_F}{C_{\dot{m}} g}$$

➢ A plot for thrust coefficient versus the area ratio $\frac{A_e}{A^*}$ for different pressure ratio $\frac{p_e}{p_{0e}}$ and a constant specific heat



Specific Propellant Consumption

Specific propellant consumption (SPC) for the rocket engine can be defined as the amount of propellant weight consumed per total impulse delivered.

$$SPC = \frac{1}{I_{sp}} = \frac{\dot{m}_p g}{F} = \frac{\dot{w}_{.p}}{F} = \frac{\dot{m}_p g}{\dot{m}_p V_{eff}} = \frac{g}{V_{eff}}$$

- **It is similar to the thrust-specific fuel consumption (TSFC) for air-breathing jet engine and brake-specific fuel consumption for automobile engines.**
- **Specific propellant consumption (SPC) is not commonly used in the rocket engine. It is basically reciprocal of specific impulse.**

Characteristic Velocity

It is defined as the ratio of equivalent exit velocity V_{eq} **and the thrust coefficient** C_F

$$C^* = \frac{V_{eq}}{C_F}$$

Or

- **Character velocity in terms of chamber pressure , nozzle throat area and propellant mass flow rate , the expression as follows,**

$$C^* = \frac{V_{eq}}{C_F} = \frac{p_c A_{th}}{\dot{m}_p} = \frac{1}{C_{\dot{m}}}$$

Or

$$C^* = \frac{1}{C_{\dot{m}}} = \frac{I_{sp} g}{C_F}$$

Or

$$I_{sp} = \frac{C^* C_F}{g}$$

Noter:

1. It represents the effectiveness with which combustion takes place in the combustion chamber of the rocket engine.
2. The characteristic velocity *C** is dependent on thrust coefficient and specific impulse.
3. It can be observed from this equation that specific impulse is dependent on two independent parameters, namely, the characteristic velocity *C** and thrust coefficient C_F .

4. The characteristic velocity, *C** represents the combustion efficiency while thrust coefficient C_F indicates the effectiveness with which high pressure hot gases are expanded in the nozzle to produce equisite thrust.

Characteristic Velocity

In terms of maximum mass flow rate of propellant (critical value,M=1)

$$\frac{\dot{m}_{max}}{A_{th}}\frac{\sqrt{T_{0e}}}{p_{0e}}\sqrt{\frac{R}{\gamma}}=\sqrt{\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Or

$$p_{0e}=\frac{\sqrt{RT_{0e}}}{\sqrt{\gamma\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma-1}}}}\left(\frac{\dot{m}}{A_{th}}\right)$$

Or

$$p_{0e}\left(\frac{A_{th}}{\dot{m}_p}\right)=\frac{\sqrt{RT_{0e}}}{\sqrt{\gamma\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Where

$$\frac{\sqrt{RT_{0e}}}{\sqrt{\gamma\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma-1}}}}=\psi$$

- ψ represents the relationship between mass flux and chamber pressure p_{0e} and is measured in m/s. This indicates the capacity of the propellant to generate a certain pressure in the combustion chamber for a given mass flow rate per unit throat area.
- It is known as the characteristic velocity *C**.

Hence

$$p_{0e}\left(\frac{A_{th}}{\dot{m}}\right)=C^*$$

qa

i.e.

$$C^*=\frac{\sqrt{RT_{0e}}}{\sqrt{\gamma\left(\frac{2}{\gamma-1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Characteristic Velocity in terms of mass flow rate is as follows

$$\dot{m}_{\max} = \dot{m} = \frac{p_{0e} A_{th}}{C^*}$$

Note

- The characteristic velocity C^* increases with adiabatic flame temperature and decreases with molecular weight of gases. It increases with decreasing value of specific heat ratio γ.
- It may be observed that the characteristic velocity C^* increases with temperature. The effect of specific heat ratio γ on C^* is more prominent at a higher temperature range compared to a lower temperature range.

Impulse-to-Weight Ratio

It is defined as the total impulse to the initial vehicle weight.

$$\frac{I}{W} = \frac{I_{sp}.m_p}{\left(m_p + m_s + m_l\right) g}$$

Where

W is the total weight of the vehicle
m_p is the propellant mass
m_s is the structural mass of the vehicle
m_l is the payload mass

Note:

1. **In order to find out the efficiency of overall design of rocket engine, it is prudent to use impulse–weight ratio.**
2. **It is desirable to have higher value of impulse–weight ratio.**

Mass Ratio (MR)

Mass ratio is defined as the final mass m_f after rocket operation (or after propellants were consumed) divided by the initial mass m_0 before rocket operation

$$MR = \frac{m_f}{m_0}$$

Propellant Mass Fraction

Propellant mass fraction is the ratio of propellant mass m_p to the initial mass m_0. It is denoted by ξ.

$$\xi = \frac{m_p}{m_0} = \frac{m_0 - m_f}{m_0} = \frac{m_p}{m_p + m_f}$$

$$\xi = \frac{m_p}{m_0} = \frac{m_0 - m_f}{m_0} = 1 - \frac{m_f}{m_0}$$

$$\xi = 1 - \frac{m_f}{m_0} = 1 - MR$$

$$\xi = 1 - MR$$

Problem

A rocket engine with chamber pressure of 4.5 MPa and nozzle throat diameter of 110 mm produces thrust of 17 kN by consuming propellant flow rate of 7.5 kg/s with calorific value of 25 MJ/kg. If the flight velocity happens to be 850 m/s, determine the specific impulse, effective exhaust velocity, specific propellant consumption and thrust power, and thrust coefficient.

Solution:

Specific impulse:

$$I_{sp} = \frac{F}{\dot{m}_p g} = \frac{17 \times 10^3}{7.5 \times 9.81} = 231.06 \text{ s}$$

Effective velocity:

$$V_{eq} = I_{sp} g = 231.06 \times 9.81 = 2266.7 \text{ m/s}$$

Specific propellant consumption:

$$SPC = \frac{1}{I_{sp}} = \frac{1}{231.06} = 0.0043 \text{ kg/N} \cdot \text{s}$$

Thrust power:

$$P_F = FV = 17 \times 850 = 14,450\,\text{kW} = 14.45\,\text{MW}$$

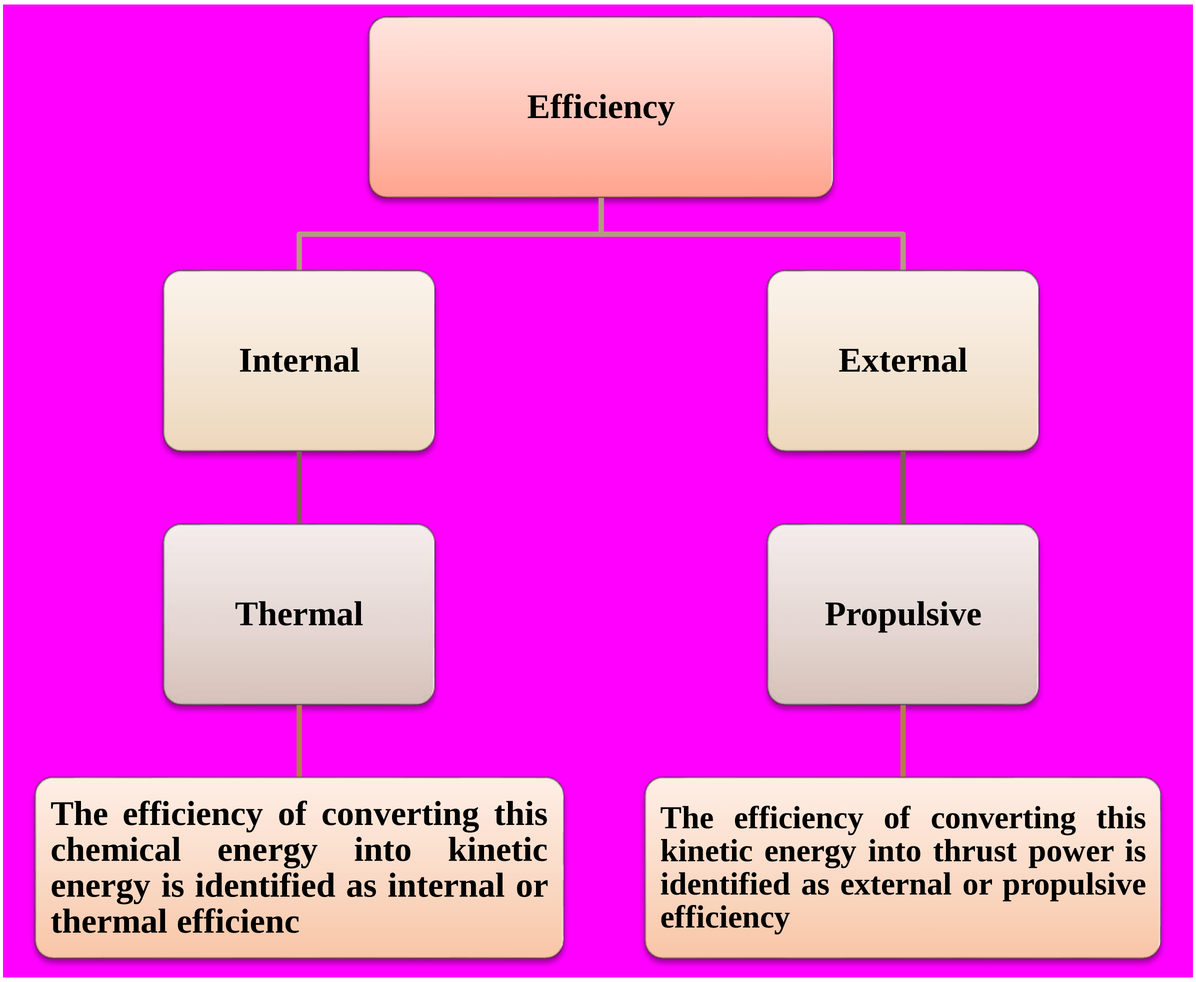
Thrust coefficient:

$$C_F = \frac{F}{P_c A_t} = \frac{4 \times 17}{2500 \times 3.14 \times (0.11)^2} = 0.72$$

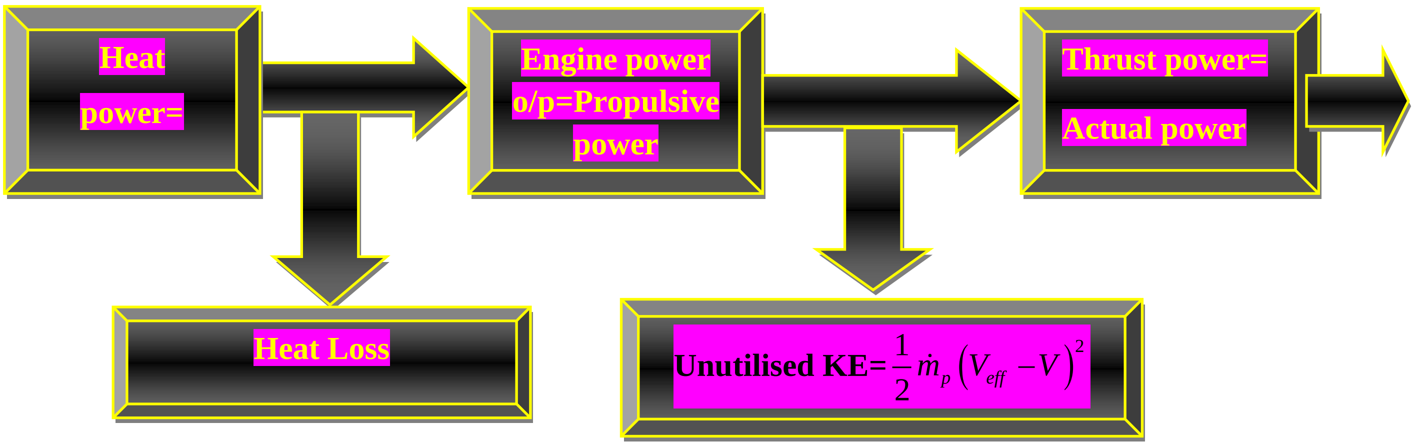
Characteristic velocity

$$C^* = \frac{V_{eq}}{C_F} = \frac{2266.7}{0.72} = 3148.2\ \text{m/s}$$

Efficiencies of Rocket engine:



Energy balance



- Thermal (or internal) efficiency of a rocket propulsion system is an indication of the effectiveness of converting the chemical power input to the propulsion device into engine output power

- Engine output power = Propulsion power or Thrust power +Power lost in exhaust gases i.e Unutilised KE

Engine output power =

$$FV+\frac{1}{2}\dot{m}_p\left(V_{eff}-V\right)^2=\dot{m}_pV_{eff}V+\frac{1}{2}\dot{m}_p\left(V_{eff}-V\right)^2=\frac{1}{2}\dot{m}_p\left(V_{eff}^2+V^2\right)$$

- The maximum energy available per unit mass of chemical propellant is the heat of the combustion $\left(HV=CV=Q_R\right)$. The power input to a chemical engine is. It can be noted that $\dot{P}_{in}$ is the total rate of energy supplied to the propulsion system. Thus, this rate of energy input $\dot{P}_{in}$ to the chemical rocket engine can be expressed as

$$\dot{P}_{in} = \dot{m}_p \times HV$$

- The actual power of combustion is obtained by multiplying the combustion efficiency η_{cc} by the power input $\dot{P}_{in}$:

So

$$\dot{P}_{in} = \eta_{cc} \times \dot{m}_p \times HV$$

Hence

Thermal efficiency

$$\eta_{th} = \frac{Engine\ power\ output}{Available\ chemical\ power\ developed\ by\ combustion}$$

$$\eta_{th} = \frac{\frac{1}{2}\dot{m}_p\left(V_{eff}^2 + V^2\right)}{\eta_{cc} \times \dot{m}_p \times HV}$$

Therefore thermal efficiency will be

$$\eta_{th} = \frac{\left(V_{eff}^2 + V^2\right)}{2\eta_{cc} \times HV}$$

Propulsive efficiency

$$\eta_p = \frac{\text{Propulsion power or thrust power}}{\text{Engine output power}} = \frac{FV}{FV + \frac{1}{2}\dot{m}_p\left(V_{eff} - V\right)^2} = \frac{\dot{m}_p V_{eff} V}{\frac{1}{2}\dot{m}_p\left(V_{eff}^2 + V^2\right)}$$

$$\eta_p = \frac{2V_{eff}V}{V_{eff}^2 + V^2} = \frac{2\left(\frac{V}{V_{eff}}\right)}{1+\left(\frac{V}{V_{eff}}\right)^2}$$

$$If \ \ \frac{V}{V_{eff}} = \sigma = Speed \ \ ratio$$

Then

$$\eta_p = \frac{2\sigma}{1+\sigma^2}$$

Remarks:

- **Propulsive efficiency can reach 100 % in rockets which is never attained in air-breathing engines.**
- **Rockets can operate with exhaust speed less than the flight speed** $\frac{V}{V_{eff}} > 1$**;**

- All air-breathing engines operate with flight speed less than exhaust speed $\frac{V}{V_{eff}} < 1$,
- As propellant material in rocket is carried within its interior; thus, it has a zero initial speed, while air-breathing engine has finite air speed coming into its core and then next accelerated to a higher value within the exhaust nozzle.

Overall efficiency

$$\eta_O = \frac{\text{Thrust Power}}{\text{Power developed by combustion}} = \frac{FV}{\eta_{cc} \times \dot{m}_p \times HV} = \frac{\dot{m}_p V_{eff} V}{\eta_{cc} \times \dot{m}_p \times HV} = \frac{V_{eff} V}{\eta_{cc} \times HV}$$

Or

$$\eta_O = \eta_{th} \times \eta_p$$

Problem:1

A rocket engine with nozzle exit diameter of 125 mm produces 7.5 kN by consuming propellant flow rate of 3.5 kg/s with calorific value of 35 MJ/kg. The chamber pressure at 6.5 MPa is expanded to exit pressure of 85 kPa. If the flight velocity happens to be 1250 m/s at altitude of 20 km, determine nozzle exhaust velocity, propulsive efficiency, overall efficiency, and thermal efficiency.

Solution:

Atmospheric pressure at 20 km altitude is 5.53kpa

Exhaust velocity:

$$V_e = \frac{F-(P_e-P_a)A_e}{\dot{m}} = \frac{7500-(85-5.53)\times 10^3 (3.14/4)(0.125)^2}{3.5}$$
$$= 1864.36 \text{ m/s}$$

Propulsive efficiency:

$$\eta_p = \frac{FV}{FV+\frac{\dot{m}_p}{2}(V_e-V)^2} = \frac{7.5\times 10^3 \times 1250}{7.5\times 10^3\times 1250+\frac{3.5}{2}(1864.36-1250)^2}$$
$$= 0.934$$

Overall efficiency:

$$\eta_o = \frac{FV}{\dot{m}_p \Delta H_P} = \frac{7.5\times10^3\times1250}{3.5\times35\times10^6} = 0.0765$$

Thermal efficiency:

$$\eta_{th} = \frac{\eta_o}{\eta_P} = \frac{0.076}{0.934} = 0.081$$

Problem:2

A rocket engine produces a thrust of 800 kN at sea level with a propellant flow rate of 300 kg/s. Calculate the specific impulse

Ans: $I_{sp} = 266.6\sec$

Problem:3

A spacecraft's engine ejects mass at a rate of 32 kg/s with an exhaust velocity of 3200 m/s. The pressure at the nozzle exit is 6 kPa and the exit area is 0.7 m^2. What is the thrust of the engine in a vacuum $(p_a = 0)$?

Ans: $F = \dot{m}_p V_e + p_e A_e$ = **106.6kN**

Problem:4

A rocket having an effective jet exhaust velocity of 1100 m/s and flying at 8500 km/h has propellant flow rate of 6 kg/s. If the heat of reaction of propellants is

43 MJ. Calculate the propulsive efficiency, thermal efficiency, and overall efficiency

Given:

$V_{eff} = 1100 m/s$

$V = 8500 km/h = 236.11 m/s$

$\dot{m}_p = 6 kg/s$

$HV = CV = Q_R = 43 MJ$

- If η_{cc} is not given take $\eta_{cc}=1$

Answer:

$$\eta_p = \frac{2\sigma}{1+\sigma^2} = 76.77\%$$

$$\eta_{th} = \frac{\left(V_{eff}^2 + V^2\right)}{2\eta_{cc} \times HV} = 7.89\%$$

$$\eta_0 = \eta_p \times \eta_{th} = 6.127\%$$

Problem:5 In a rocket engine with nozzle exit diameter of 105 mm, hot gas at 2.5 MPa is expanded to exit pressure and temperature of 85 kPa and 1200 K, respectively. If the mass flow rate happens to be 75 kg/s, determine the exit jet velocity, effective jet velocity, and thrust at an altitude of 25 km. Take calorific value of propellant as 22 MJ/kg.

Given:

$$p_{0e} = p_c = 2.5 Mpa, p_e = 85 kpa, T_e = 1200 K$$
$$\dot{m}_p = 75 kg/s, HV = 22 MJ/kg, p_{a/25km} = 5.45 kp$$

Hint: $\dot{m}_p = \rho A_e V_e$

Where $\rho = \dfrac{p}{RT}$

Problem:6

The hot propellant gas at chamber pressure of 3.5 MPa with a flow rate of 5.5 kg/s is expanded fully through a CD nozzle with throat diameter of 80 mm to produce thrust of 12 kN. If the flight velocity happens to be 750 m/s, determine the specific impulse, effective exhaust velocity, specific propellant consumption and thrust power, and thrust coefficient. Take calorific value of propellant as 22 MJ/kg.

Hint:

$$1.\ I_{sp} = \frac{I}{m_p g} = \frac{m_p V_{eq}}{m_p g} = \frac{V_{eq}}{g} = \frac{F}{\dot{m}_p g} = \frac{F}{\dot{w}_p} = 222.634\sec$$

$$2.\ V_e = \frac{F}{\dot{m}}$$

$$V_e = V_{eff} = \frac{F}{\dot{m}} = 2181.81 m/s$$

$$3.\ SPC = \frac{1}{I_{sp}} = \frac{\dot{m}_p g}{F} = \frac{\dot{w}_{.p}}{F} = \frac{\dot{m}_p g}{\dot{m}_p V_{eff}} = \frac{g}{V_{eff}} = 0.0045$$

4. Thrust power= FXV=12X750=9000KW

$$5.\ C_F = \frac{F}{p_c\left(A_{th} = A^*\right)} = 0.6824$$

Problem:7

A rocket nozzle has a throat area of 18 cm^2 and combustion chamber pressure of 25 bar. If the specific impulse is 127.42 s and weight flow rate is 44.145 N/s, determine the thrust coefficient, propellant mass flow coefficient, specific propellant consumption, and characteristic velocity.

Solution:

$$1.\ C_F = \frac{F}{p_c\left(A_{th} = A^*\right)} = \frac{I_{sp}\dot{m}_p g}{p_c\left(A_{th} = A^*\right)} = \frac{I_{sp}\dot{w}_p}{p_c\left(A_{th} = A^*\right)} = 1.233$$

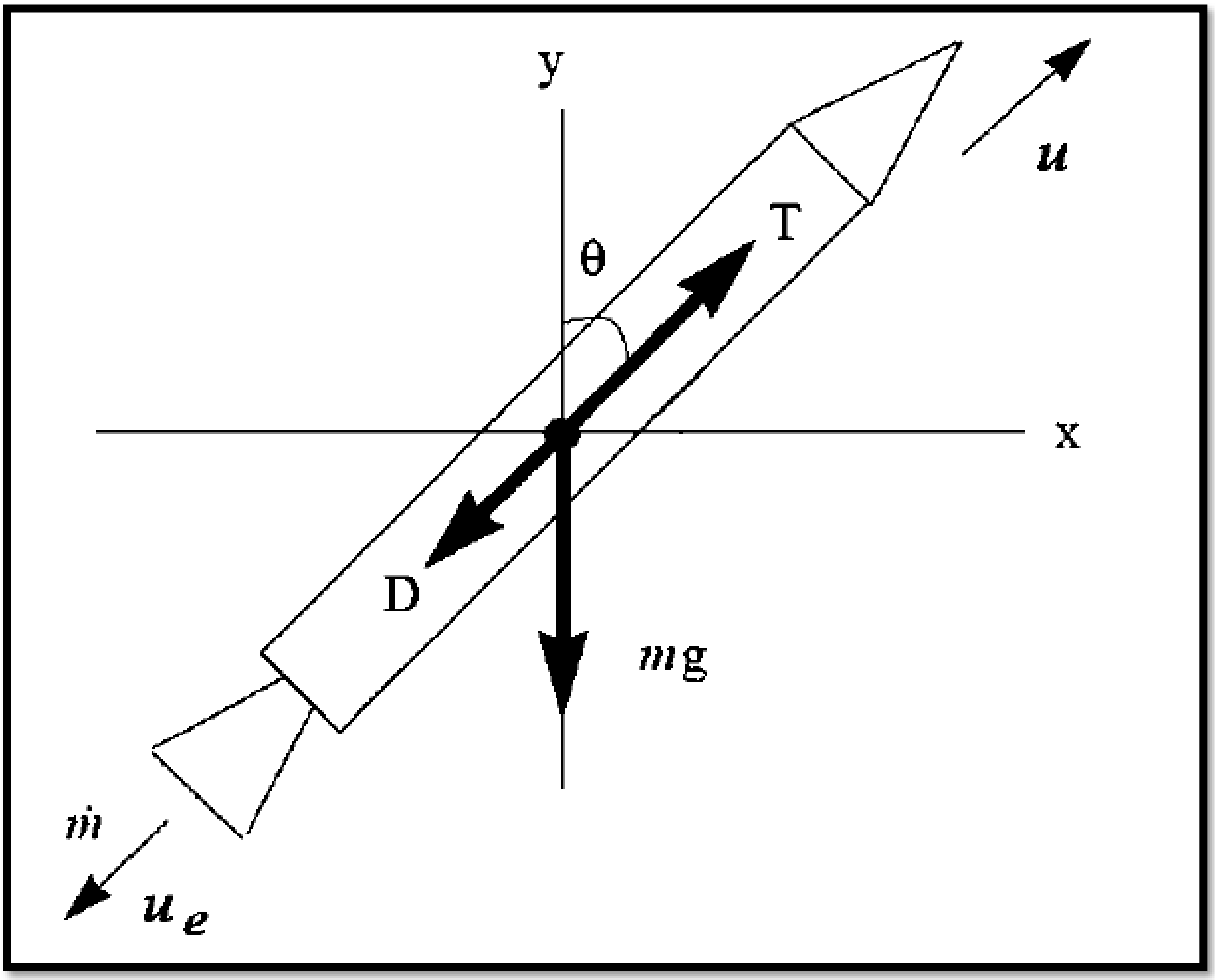
$$2.\ C_{\dot{m}} = \frac{\dot{m}_p}{p_c A_t} = 0.001$$

$$3.\ SPC = \frac{1}{I_{sp}} = \frac{\dot{m}_p g}{F} = \frac{\dot{w}_{.p}}{F} = \frac{\dot{m}_p g}{\dot{m}_p V_{eff}} = \frac{g}{V_{eff}} = 0.00785$$

$$4.\ C^* = \frac{V_{eq}}{C_F} = \frac{p_c A_{th}}{\dot{m}_p} = \frac{1}{C_{\dot{m}}} = 1000 m/s$$

Rocket Equation

- **Let us assume the flight trajectory is two-dimensional and is constrained in the *x*–*y* plane as shown following figure**
- **However, we will consider the aerodynamic forces, namely, drag and lift, that are acting on the vehicle along with gravitational force from the earth.**
- **Note that we assume that the direction of flight is the same as that of the thrust The equation of motion of a rocket is derived here.**
- **Newton's second law of motion is used. The forces governing the motion of a rocket are thrust (T), drag (D), lift (L), and gravitational force (mg). Here (m) is the instantaneous rocket mass.**
- **First it is assumed that both thrust and drag are aligned in the flight direction. Flight direction is inclined an angle (θ) to the vertical direction**



- The forces acting on the rocket may be resolved in the flight direction to give

$$m\frac{dV}{dt} = F - D - mg\cos\theta$$

- The thrust force in the above equation can be either expressed as

$$F = \dot{m}_p V_{eff}$$

$$or$$

$$F = \dot{m}_p g I_{sp}$$

$$m\frac{dV}{dt} = \dot{m}_p V_{eff} - D - mgcos\theta$$

- Noting that a large fraction (typically 90 %) of the mass of a rocket is propellant, thus it is important to consider the change in mass of the vehicle as it accelerates. Since the rate of mass flow through the nozzle is equal to the negative rate of change of rocket mass, thus

$$\dot{m} = -\frac{dm}{dt}$$

$$m\frac{dV}{dt} = -\frac{dm_p}{dt}V_{eff} - D - mgcos\theta$$

- We know that the vehicle mass is reduced as the propellant is ejected through the nozzle of the rocket engine for producing thrust. Hence, the instantaneous mass of the vehicle can be related to the propellant mass as follows

$$m = m_0 - \dot{m}_p t = m_0 - \frac{m_p}{t_b}t = m_0\left(1 - \left(\frac{m_p}{m_0}\right)\frac{t}{t_b}\right) = m_0\left(1 - PF\frac{t}{t_b}\right)$$

Where

t is the instantaneous time

t_b is the total burning time

$\dot{m}_p$ is the initial propellant mass

$\dot{m}_0$ is the initial mass of the vehicle

PF = $\frac{m_p}{m_0} = \xi$ is the propellant fraction

$$m_0\left(1-PF\frac{t}{t_{tb}}\right)\frac{dV}{dt} = \frac{m_p}{t_b}V_{eq} - D - m_0\left(1-PF\frac{t}{t_{tb}}\right)g\cos\theta$$

Or

$$\frac{dV}{dt} = \frac{m_p V_{eq}}{t_b\left[m_0\left(1-PF\frac{t}{t_b}\right)\right]} - \frac{D}{\left[m_0\left(1-PF\frac{t}{t_b}\right)\right]} - g\cos\theta$$

Or

$$\frac{dV}{dt} = \frac{\left(\frac{PF}{t_b}\right)V_{eq}}{\left[\left(1-PF\frac{t}{t_b}\right)\right]} - \frac{D}{\left[m_0\left(1-PF\frac{t}{t_b}\right)\right]} - g\cos\theta$$

Drag loss   Gravity loss

- This equation is often known as the rocket equation

Case: 1 Negligible drag

$$\frac{dV}{dt} = \frac{\left(\frac{PF}{t_b}\right) V_{eq}}{\left[\left(1 - PF\,\frac{t}{t_{tb}}\right)\right]} - g\cos\theta$$

- Integrating the above equation over a period of time equal to the burning time t_b to get the increment change in rocket velocity $\left(\Delta V = V_b - V_0\right)$
- Integration boundary condition as follows

$$t = 0, V = V_0$$

$$and$$

$$t = t_b, V = V_b$$

$$\int_{V_0}^{V_b} dV = \left(\frac{PF}{t_b}\right) V_{eq} \int_0^{t_b} \frac{dt}{\left[\left(1 - PF\,\frac{t}{t_b}\right)\right]} - g\cos\theta \int_0^{t_b} dt$$

$$\Delta V = V_b - V_0 = -V_{eq} \ln\left[\frac{1}{1 - PF}\right] - \left(g\cos\theta\right) t_b$$

- The propellant fraction PF can be expressed in terms of mass ratio MR as follows

$$\frac{1}{1-PF}=\frac{1}{1-\frac{m_p}{m_0}}=\frac{m_0}{m_0-m_p}=\frac{m_0}{m_b}=\frac{m_0}{m_f}=\frac{1}{MR}$$

where

- m_b is the burnout mass of the vehicle after all the propellant is consumed
- *MR* is the mass ratio of the vehicle

$$\Delta V=V_b-V_0=V_{eq}\ln\left(\frac{1}{MR}\right)-\left(g\cos\theta\right)t_b$$

Or

In terms of specific impulse

$$\Delta V=V_b-V_0=I_{sp}g\ln\left(\frac{1}{MR}\right)-\left(g\cos\theta\right)t_b$$

- For larger Δ*V*, we need to have higher I_{sp} and *MR* and short burning duration when the vehicle is travelling through the gravitational field of any planet. As the time is too short to have a larger Δ*V* for the same I_{sp} and *MR*, the vehicle must be impulsive in nature.

Case 2: Constant gravitational acceleration and Horizontal flight $(\theta=0)$

$$\Delta V=V_b-V_0=V_{eq}\ln\left(\frac{1}{MR}\right)-gt_b=I_{sp}g\ln\left(\frac{1}{MR}\right)-gt_b$$

$$\Delta V = \left(I_{sp} \ln \frac{1}{MR} - t_b \right) g$$

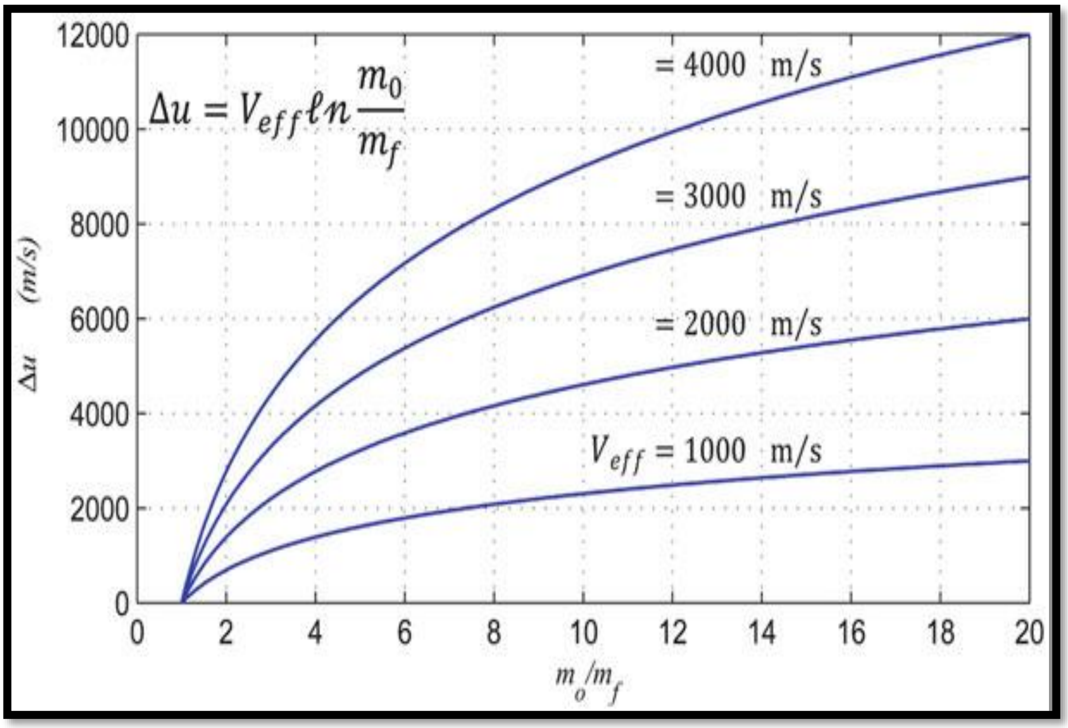
Case 3: If the vehicle is travelling in space, then the change in velocity due to gravitational force will be almost zero, which is called gravity free flight.

$$\Delta V = V_b - V_0 = V_{eq} \ln\left(\frac{1}{MR}\right) = I_{sp} g \ln\left(\frac{1}{MR}\right)$$

Hence for negligible drag and gravity loss, increment change in rocket velocity ΔV

$$\Delta V = V_b - V_0 = V_{eq} \ln\left(\frac{1}{MR}\right) = I_{sp} g \ln\left(\frac{1}{MR}\right)$$

- Known as Tsiolkovsky rocket equation.



A plot of Tsiolkovsky for different effective speeds

$$\Delta V = V_b - V_0 = V_{eq} \ln\left(\frac{1}{MR}\right) = V_{eq} \ln\left(\frac{m_0}{m_f}\right)$$

$$\ln\left(\frac{m_0}{m_f}\right) = \frac{\Delta V}{V_{eq}}$$

$$\frac{m_0}{m_f} = e^{\frac{\Delta V}{V_{eq}}}$$

$$m_0 = m_f e^{\frac{\Delta V}{V_{eq}}}$$

$$m_0 = \left(m_0 - m_p\right) e^{\frac{\Delta V}{V_{eq}}} = \left(m_0 - \dot{m}t\right) e^{\frac{\Delta V}{V_{eq}}}$$

Hence

Burning time:

$$t_b = \frac{m_0}{\dot{m}}\left(1 - \frac{1}{e^{\frac{\Delta V}{V_{eff}}}}\right)$$

Payload ratio or payload fraction :

It is defined as the ratio of payload mass to initial mass of rocket. It is denoted by λ

$$\lambda = \frac{m_L}{m_0}$$

m_L : It is the payload mass (which may be a satellite, space probe, or astronauts).

Structural (dead) weight ratio or structural fraction

It is defined as the ratio of structural mass to initial mass of rocket. It is denoted by δ

$$\delta = \frac{m_S}{m_0}$$

m_S: It is the structural mass of the rocket if single stage or (i_{th}) stage alone including the mass of its engine, controllers, and instrumentation as well as any residual propellant which is not expended by the end of the burn. So initial mass of rocket is given as follows:

$$m_0 = m_L + m_S + m_P$$

Dividing by m_0 both side

Hence

$$\frac{m_0}{m_0} = \frac{m_L}{m_0} + \frac{m_S}{m_0} + \frac{m_p}{m_0}$$

$$1 = \lambda + \delta + \xi$$

OR

$$1 - \xi = \lambda + \delta = MR$$

Now we know that

$$\frac{m_0}{m_0 - m_p} = \frac{m_0}{m_b} = \frac{m_0}{m_f} = \frac{1}{MR}$$

Here as we know

$$m_b = m_f = m_0 - m_p = m_L + m_S$$

From above expression

$$\frac{m_0}{m_0 - m_p} = \frac{m_0}{m_L + m_S} = \frac{1}{\lambda + \delta} = \frac{1}{MR}$$

Now putting this value in above equation of rocket

$$\Delta V = V_b - V_0 = V_{eq} \ln\left(\frac{1}{\lambda+\delta}\right) = I_{sp} g \ln\left(\frac{1}{\lambda+\delta}\right)$$

Structural coefficient:

It is defined as the ratio of structural mass to the sum of structural and propellant masses, it is denoted by ε.

$$\varepsilon = \frac{m_S}{m_p + m_S} = \frac{m_b - m_L}{m_0 - m_L}$$

This expression is true only when the entire propellant is burnt out without any residual unburnt propellant mass. It is desirable to have a smaller value of structural coefficient for space applications because the smaller is its value, the lighter will be the vehicle. In other words, the structural coefficient indicates how far the designer can manage to reduce the structural mass.

By using the two mass ratios, namely, payload ratio λ and structural coefficient ε

$$\frac{1}{MR} = \frac{m_0}{m_f} = \frac{m_0}{m_b} = \frac{m_0}{m_0 - m_p} = \frac{1+\lambda}{\varepsilon+\lambda}$$

Hence, Incremental velocity ΔV is given as follows

$$\Delta V = V_b - V_0 = V_{eq} \ln\left(\frac{1}{MR}\right) = V_{eq} \ln\left(\frac{m_0}{m_f}\right)$$

$$\Delta V = V_{eq}\, ln\left(\frac{1+\lambda}{\varepsilon+\lambda}\right)$$

Problem 1:

A spacecraft's dry mass is 75,000 kg and the effective exhaust gas velocity of its main engine is 3100 m/s. How much propellant must be carried if the propulsion system is to produce a total ΔV of 700 m/s?

Solution:

$$\Delta u = V_{\text{eff}} \ln \frac{m_0}{m_f}$$

$$m_0 = m_f e^{\frac{\Delta u}{V_{\text{eff}}}} = 75,000 \times e^{\frac{700}{3,100}} = 94,000\,\text{kg}$$

The propellant mass is $m_p = m_0 - m_f = 94,000 - 75,000 = 19,000$ kg.

Problem 2:

A rocket vehicle has the following data:
Initial mass = 300 kg
Final mass = 180 kg
Payload mass = 130 kg
Burn duration = 5 s
Specific impulse = 280 s
Determine the mass ratio, propellant mass fraction, propellant flow rate, thrust-to-weight ratio, and impulse-to-weight ratio of the vehicle.
Given:

$$m_0 = 300kg, m_f = 180kg, m_L = 130kg, t_b = 5\sec, I_{sp} = 280\sec$$

Solution:

$$MR = \frac{m_f}{m_o} = 0.6$$

$$\xi = PF = \frac{m_P}{m_o} = \frac{m_o - m_f}{m_o} = 0.4$$

$$\frac{m_p}{t_b} = 24kg/s$$

$$Thrust \quad F = I_{sp}\dot{m}_p g = 280 \times 24 \times 10$$

$$= 67200N$$

$$Wt. = m_0 g = 3000N$$

$$\frac{T}{W} = 224$$

$$\frac{I_{sp}}{W} = 0.093$$

Problem 3:

A single-stage chemical rocket with I_{sp} = 250sec is designed to escape with the following mass: m_L = payload mass = 200 kg; m_S = structural mass = 800 kg; m_0 = total mass = 30,000 kg. Determine the mass ratio, velocity increment, payload, and structural fraction for this rocket engine assuming there are no drag and gravity effects

Given: I_{sp} = 250, m_L = payload mass = 200 kg; m_S = structural mass = 800 kg; m_0 = total mass = 30,000 kg

Solution:

$$MR = \frac{m_f}{m_0} = \frac{m_0 - m_p}{m_0} = \frac{m_s + m_L}{m_0} = 0.033$$

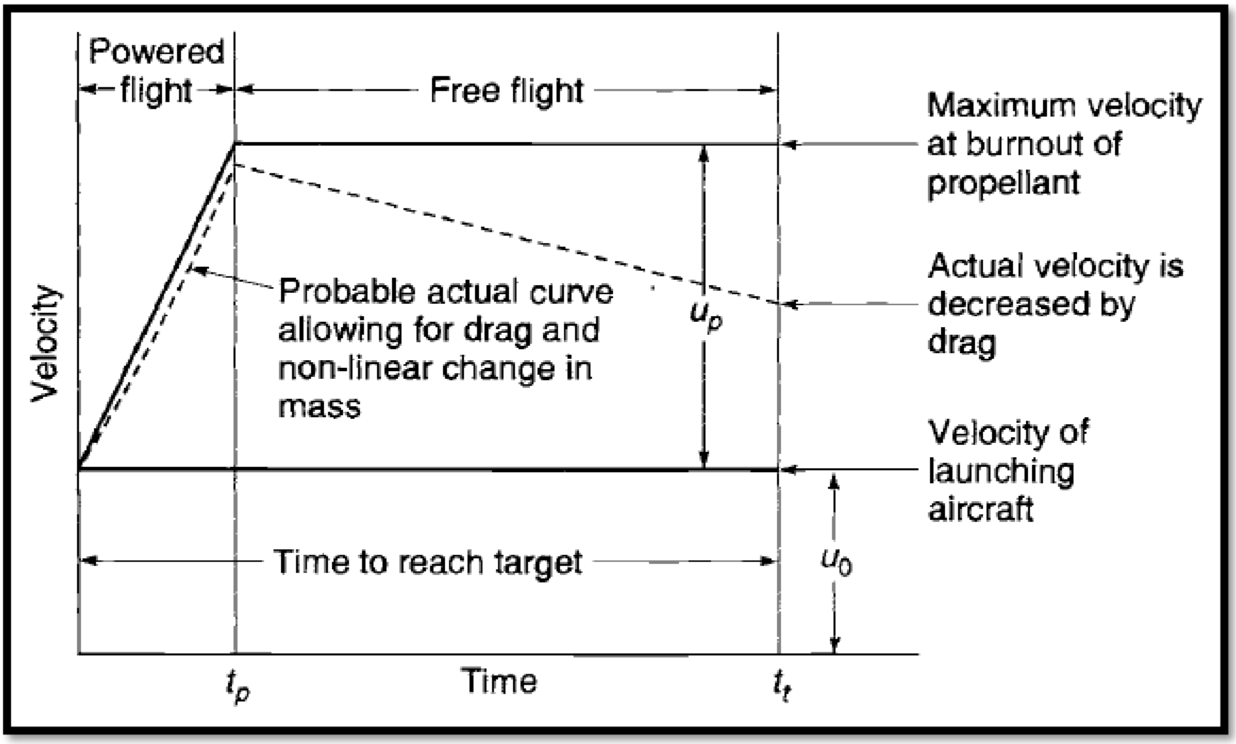
$$\Delta V = V_{eff} \ln\left(\frac{m_0}{m_f}\right) = I_{SP} \times g \ln\left(\frac{m_0}{m_f}\right) = 8502.99 m/s$$

$$\lambda = \frac{m_L}{m_0} = Payload = 0.0066$$

$$\delta = \frac{m_S}{m_0} = 0.026 = structural$$

Flight Trajectory:

We will consider the flight trajectory as 2-D in nature which will be on a single plane. This is possible only when the vehicle is not affected by the wind, thrust misalignment, solar attraction, and so on. In this case, the flight direction angle is the same as the thrust direction angle. The lift force can be assumed to be negligible as it is a wingless and symmetrical rocket engine vehicle



$t_0 = t_t$ =Total or overall or complete time of flight

$t_p = t_b$ = Powered flight time or burn time

$t_c = t_t - t_p$ = Coasting flight time

Assumption:

- Flight is vertical
- Aerodynamic drag is zero
- Acceleration is constant

Powered Flight:

The powered flight takes place during burn out time t_p achieving an altitude H_p and velocity U_p. After burn out the rocket coasts to zero upward velocity reaching an altitude $H_{\max}$. After this the rocket starts falling down.

Lets initial mass of rocket during lift off is m_0

Hence by force balance (Newton's 2nd law) in Vertical direction

$$m_0 a = F - m_0 g$$

Rocket acceleration during power flight

$$a = \frac{F}{m_0} - g \text{ ----(1)}$$

Now from Velocity-time equation of motion

$$V = U + at \text{ ----(2)}$$

Apply BC's at initial and final condition of flight

$U = 0$,During lift off condition i.e initial condition

$V = U_P$, At maximum vertical hight covered by rocket

So From equation (1)&(2)

$$U_P = at_p = \left(\frac{F}{m_0} - g\right)t_p$$ ——This is velocity gain by rocket at the end of powered flight

Altitude (H_P) attained in power flight is given by using Distance-acceleration equation of motion

$$H_P = U_P t_p + \frac{1}{2}at_p^2$$ here U_P=0

Hence,

$$H_P = \frac{1}{2}at_p^2 \quad \text{——(3)}$$

By putting value of a from equation(1) in above equation (3)

$$(H_P) = \frac{1}{2}at_p^2 = \frac{1}{2}\left(\frac{F}{m_0} - g\right)t_p^2 = \frac{1}{2}\frac{F}{m_0}t_p^2 - \frac{1}{2}gt_p^2$$

Here

$\frac{1}{2}gt_p^2$: **It is called as altitude loss due to gravity.**

$$\left(H_P\right)=\frac{1}{2}at_p^2=\frac{1}{2}\left(\frac{F}{m_0}-g\right)t_p^2=\frac{1}{2}\frac{F}{m_0}t_p^2-\frac{1}{2}gt_p^2$$

Putting the value of thrust force in above equation

$$F=\dot{m}_pV_{eff}=\dot{m}_p\times I_{sp}\times g$$

Here the effective velocity is nothing but maximum velocity at the end of the powered flight which is denoted by U_P here

Hence

$$F=\dot{m}_pV_{eff}=\dot{m}_p\Delta V=\dot{m}_p\left(V_b-V_0=V_{eq}\ln\left(\frac{1}{MR}\right)-gt_b=I_{sp}g\ln\left(\frac{1}{MR}\right)-gt_b\right)$$

$$F=\dot{m}_p\left(V_{eq}\ln\left(\frac{1}{MR}\right)-gt_b\right)=\dot{m}_p\left(I_{sp}g\ln\left(\frac{1}{MR}\right)-gt_b\right)$$

Coasting Height:

Once the entire propellant of a rocket engine is burnt out, the rocket will travel further till all its kinetic energy is consumed by the vehicle. This vertical height achieved by the rocket engine is commonly known as the coasting height. By neglecting the drag and gravitational forces on the vehicle, the coasting height H_c can be determined by equating the kinetic energy possessed by the vehicle at burnout to the increase in potential energy due to the gain in vertical height

Remarks

- Thrust is zero
- It reaches maximum altitude H_{max}
- Decelerate to zero velocity

Now using velocity-time equation of motion

$$V = U + at$$

Here at maximum altitude final velocity becomes zero $V = 0$ and initial velocity will be the same as final velocity during power flight i.e. $U = U_P$

Now above equation becomes after putting the initial and final value of velocity

$$0 = U_P - gt_c$$

Now flight time for Coasting is given as follows

$$t_c = \frac{U_P}{g} \text{ ---(4)}$$

Altitude gain by rocket during coasting flight it is calculated by distance –time equation of motion

$$H_c = U_P t_c - \frac{1}{2} g t_c^2$$ **Putting the value of** t_c **from equation (4)**

Hence

$$H_c = U_P t_c - \frac{1}{2} g t_c^2 = U_P \times \frac{U_P}{g} - \frac{1}{2} g \times \left(\frac{U_P}{g}\right)^2 = \frac{U_P^2}{g} - \frac{1}{2}\frac{U_P^2}{g} = \frac{U_P^2}{2g} \text{ -----(5)}$$

Putting the value of U_P **in above equation (5)**

$$U_P = at_p = \left(\frac{F}{m_0} - g\right) t_p$$

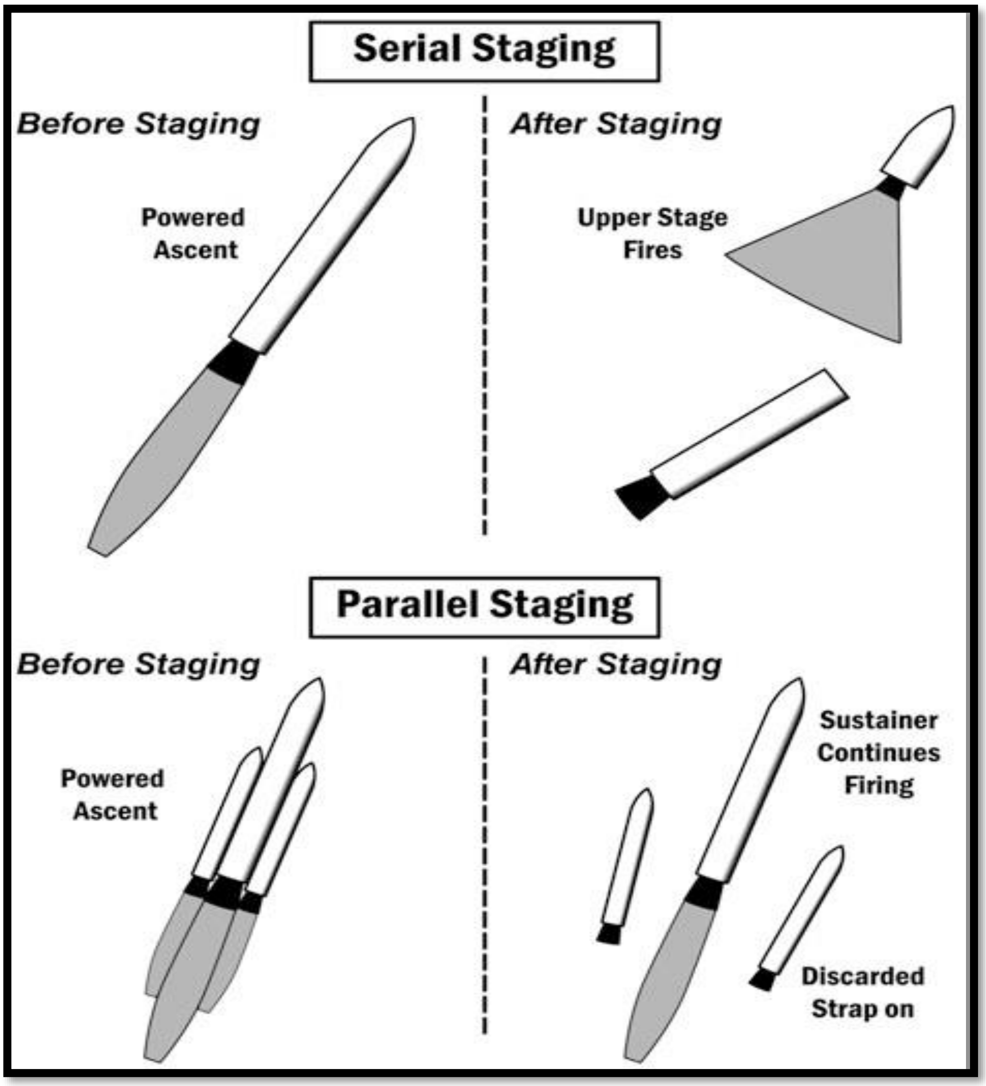
$$H_c = \frac{U_P^2}{2g} = \frac{1}{2g}\left[\frac{F}{m_0} - g\right]^2 t_p^2 \text{ -----(6)}$$

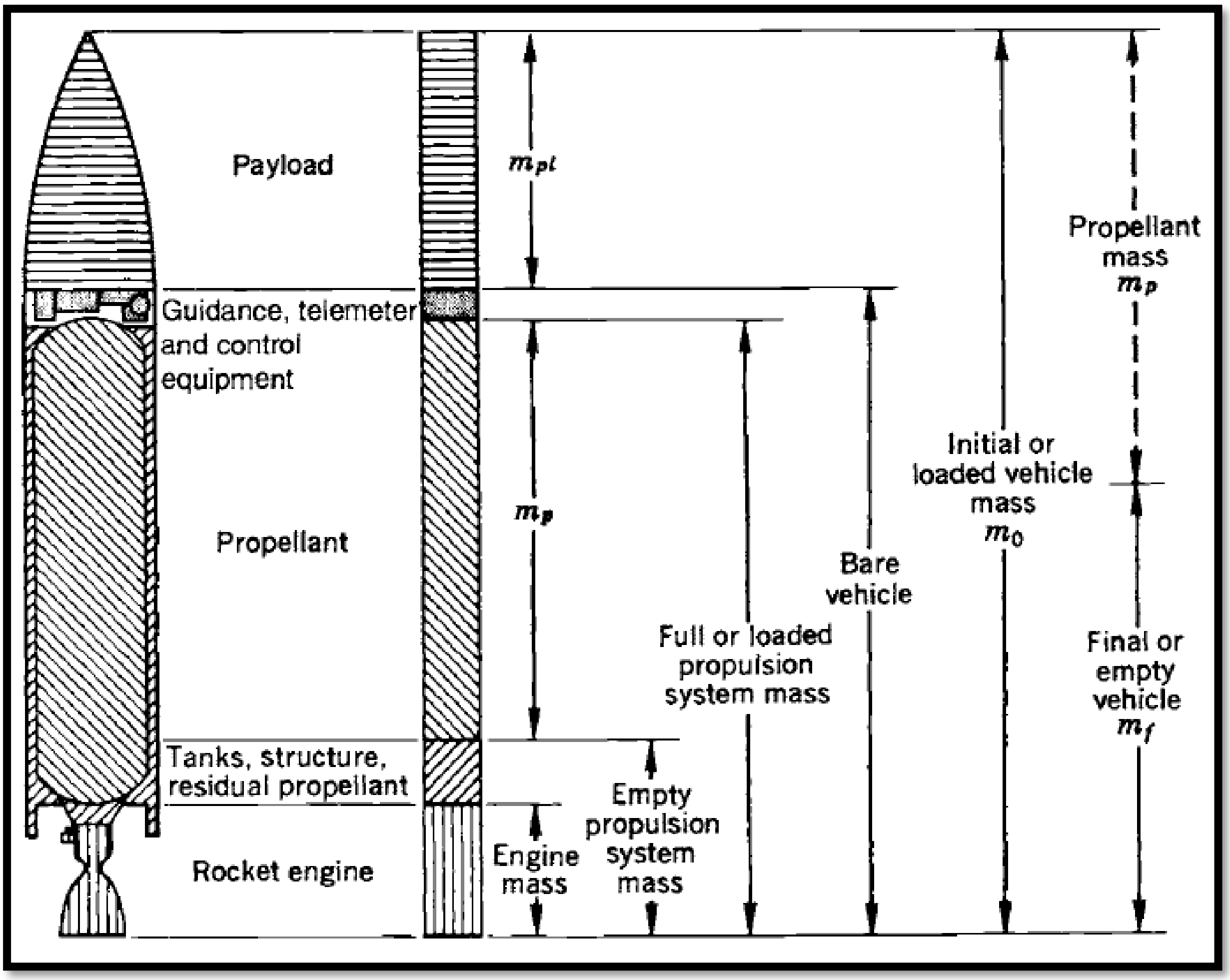
Now total altitude during powered and coasting flight is as given follows

$$H = H_p + H_c$$

$$H = \frac{1}{2}\left(\frac{F}{m_0} - g\right)t_p^2 + \frac{1}{2g}\left[\frac{F}{m_0} - g\right]^2 t_p^2$$

Staging of Rocket:




Payload
m_{pl}
Guidance, telemeter and control equipment
Propellant
m_p
Tanks, structure, residual propellant
Rocket engine
Engine mass
Empty propulsion system mass
Full or loaded propulsion system mass
Bare vehicle
Initial or loaded vehicle mass
m_0
Propellant mass
m_p
Final or empty vehicle
m_f

Rocket Equation for a Series Multistage Rocket

Instead of a single large rocket, a series of rocket motors each with its own structure, tanks, and engines are used to enhance the velocity increment for the entire vehicle for the same I_{SP} If i refer to the i_{th} of a total of N stages, then the total change in velocity is

$$\Delta V=\sum_{i=1}^{N}\Delta V_i=\sum_{i=1}^{N}\left(V_{eff}\right)_i\times\ln\left(\frac{m_0}{m_f}\right)_i=\sum_{i=1}^{N}\left(V_{eff}\right)_i\times\ln\left(\frac{1}{MR}\right)_i=g\sum_{i=1}^{N}\left(I_{SP}\right)_i\times\ln\left(\frac{1}{MR}\right)_i$$

$$\Delta V=\sum_{i=1}^{N}\Delta V_i=\sum_{i=1}^{N}\left(V_{eff}\right)_i\times\ln\left(\frac{1}{\lambda_i+\delta_i}\right)$$

$$\Delta V=\sum_{i=1}^{N}\Delta V_i=g\sum_{i=1}^{N}\left(I_{SP}\right)_i\times\ln\left(\frac{1}{\lambda_i+\delta_i}\right)$$

When the effective speed $\left(V_{eff}\right)$ and mass ratios are the same for all the stages

$$\Delta V=N\times V_{eff}\times\ln\left(\frac{1}{MR}\right)$$

Moreover, the payload of the i_{th} stage is the sum of all the succeeding stages. Thus, the overall payload ratio $\left(\lambda_0\right)$ is equal to the product of all payloads $\left(\lambda_i\right)$. Thus,

$$\left(\lambda_0\right)=\prod_{i=1}^{N}\lambda_i$$

Rocket Equation for a Parallel Multistage Rocket:

For (k) stages, we have the total thrust as the sum of all engines

$$T_t = \sum_{i=1}^{k} \dot{m}_i \left(V_{eff}\right)_i$$

Total mass flow is

$$\dot{m}_t = \sum_{i=1}^{k} \dot{m}_i$$

Thrust force is then

$$T_t = \dot{m}_t \left(V_{eff}\right)_{avg}$$

Where $\left(V_{eff}\right)_{avg}$ is the average exhaust speed and given by

$$\left(V_{eff}\right)_{avg} = \frac{T_t}{\dot{m}_t} = \frac{\sum_{i=1}^{k} \dot{m}_i \left(V_{eff}\right)}{\sum_{i=1}^{k} \dot{m}_i}$$

The rocket equation is then

$$\Delta V = \left(V_{eff}\right)_{avg} \times \ln\left(\frac{m_0}{m_f}\right)$$

Problems:

A 3-stages rocket is designed to place a small satellite into the low Earth orbit, having a payload weight of 200 kg . Data for the rocket is given in Table below. Calculate the increase in its speed ΔV .

Stage Number	Specific Impulse(sec)	Total mass (kg)	Propellant mass (kg)
1	230	14,000	12,800
2	250	4500	3700
3	250	1000	700

Solution:

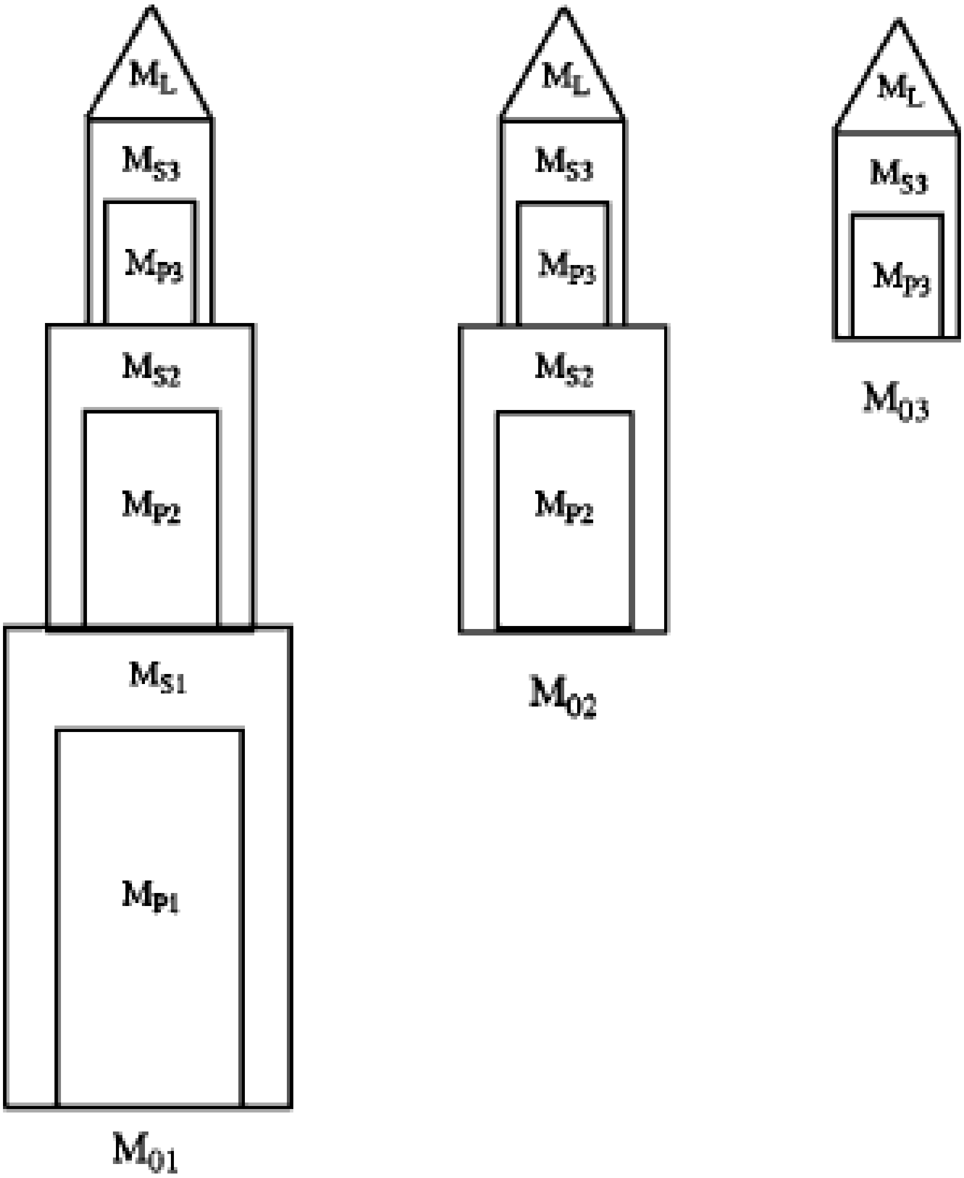
For 3 stages hence $N = 3$

HINT-1

$$\Delta V = \sum_{i=1}^{3} \Delta V_i = \Delta V_1 + \Delta V_2 + \Delta V_3$$

HINT-2

$$\Delta V = \sum_{i=1}^{N} \Delta V_i = g \sum_{i=1}^{N} \left(I_{SP} \right)_i \times \ln \left(\frac{m_0}{m_0 - m_p} \right)_i$$



M_L
M_{S3}
M_{P3}
M_{S2}
M_{P2}
M_{S1}
M_{P1}
M_{01}
M_L
M_{S3}
M_{P3}
M_{S2}
M_{P2}
M_{02}
M_L
M_{S3}
M_{P3}
M_{03}

Change in Velocity for N stages can be calculated one by one:

$$\Delta V = \sum_{i=1}^{N} \Delta V_i = g \sum_{i=1}^{N} (I_{SP})_i \times ln\left(\frac{m_0}{m_0 - m_p}\right)_i$$

First stage:

Since the payload is carried a top the third stage into orbit, the total mass of the first stage is

$$m_{01} = m_{t1} + m_{t2} + m_{t3} + m_L$$

$$m_{01} = 14{,}000 + 4500 + 1000 + 200 = 19{,}700\,\text{kg}$$

$$m_{01} - m_{p1} = 19{,}700 - 12{,}800 = 6900\,\text{kg}$$

$$(I_{sp})_1 \times \ln\left(\frac{m_{01}}{m_{01} - m_{p1}}\right) = 285 \times \ln\left(\frac{19{,}700}{6900}\right) = 285 \times 1.049 = 299\,\text{s}$$

$$\Delta V_1 = 299 Sec$$

Second stage:

As the propellant of the first stage is burned off during powered ascent, the near empty tank and structure of the first stage are dropped off to reduce the weight of the vehicle to achieve orbital velocity. Smaller amount of propellant is retained in the second- and third-stage tanks

$$m_{02} = m_{t2} + m_{t3} + m_L$$

$$m_{02} = 4500 + 1000 + 200 = 5700\,\text{kg}$$

$$m_{02} - m_{p2} = 5700 - 3700 = 2000\,\text{kg}$$

$$(I_{sp})_2 \times \ln\left(\frac{m_{02}}{m_{02} - m_{p2}}\right) = 265 \times \ln\left(\frac{5700}{2000}\right) = 265 \times 1.0473 = 277.5\,\text{s}$$

$$\Delta V_2 = 277.5 Sec$$

Third stage:

The second stage is dropped off also; thus,

$$m_{03} = m_{t3} + m_L$$

$$m_{03} = 1000 + 200 = 1200\,\text{kg}$$

$$m_{03} - m_{p3} = 1,200 - 700 = 500\,\text{kg}$$

$$(I_{sp})_3 \times \ln\left(\frac{m_{03}}{m_{03} - m_{p3}}\right) = 290 \times \ln\left(\frac{1200}{500}\right) = 290 \times 0.875 = 253.8\,\text{s}$$

$$\Delta V_3 = 253.8 Sec$$

Total incremental Velocity

$$\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$$

$$\Delta V = \sum_{i=1}^{3} \Delta V_i = g \sum_{i=1}^{3} (I_{SP})_i \times \ln\left(\frac{m_0}{m_0 - m_p}\right)_i$$

$$\Delta V = 9.8(299 + 275.5 + 253.8) = 8152.8\ m/s$$

Problem:

The shown figure illustrates European rocket Ariane 5, used to deliver payloads into the geostationary transfer orbit (GTO) or low Earth orbit (LEO). Initially at liftoff of an Ariane 5 launch vehicle, two P230 solid-propellant boosters plus the main Vulcain engine are ignited. The two different components of the launch vehicle have the following characteristics:

Engine	Vulcain (Main engine)	Booster P230
Type	Liquid Propellant	Solid propellant
Thrust(kN)	1114	6470
Specific impulse (Sec)	430	275

Calculate:

1. Effective velocity for each engine
2. Exhaust mass flow rate
3. Average effective exhaust velocity
4. Average specific impulse

Solution:

As described previously, Ariane 5 has two boosters (stage 0) employing solid propellant engines and a liquid-propellant engine. Thus, the propulsion system here resembles parallel staging

1. Effective velocity

$$I_{sp} = \frac{V_{\text{eff}}}{g}$$

$$V_{\text{eff}} = gI_{sp}$$

For the main engine (Vulcain),

$$(V_{\text{eff}})_{\text{Vulcain}} = 9.81 \times 430 = 4,218\,\text{m/s}$$

For Booster (P230),

$$(V_{\text{eff}})_{\text{P230}} = 9.81 \times 275 = 2,698\ \text{m/s}$$

2. Exhaust mass flow rate:

$$\dot{m} = \frac{T}{V_{\text{eff}}}$$

For the main engine (Vulcain),

$$(\dot{m})_{\text{Vulcain}} = \frac{T}{V_{\text{eff}}} = 264\,\text{kg/s}$$

For Booster (P230),

$$(\dot{m})_{\text{P230}} = \frac{T}{V_{\text{eff}}} = 2398\,\text{kg/s}$$

3. Average effective exhaust velocity

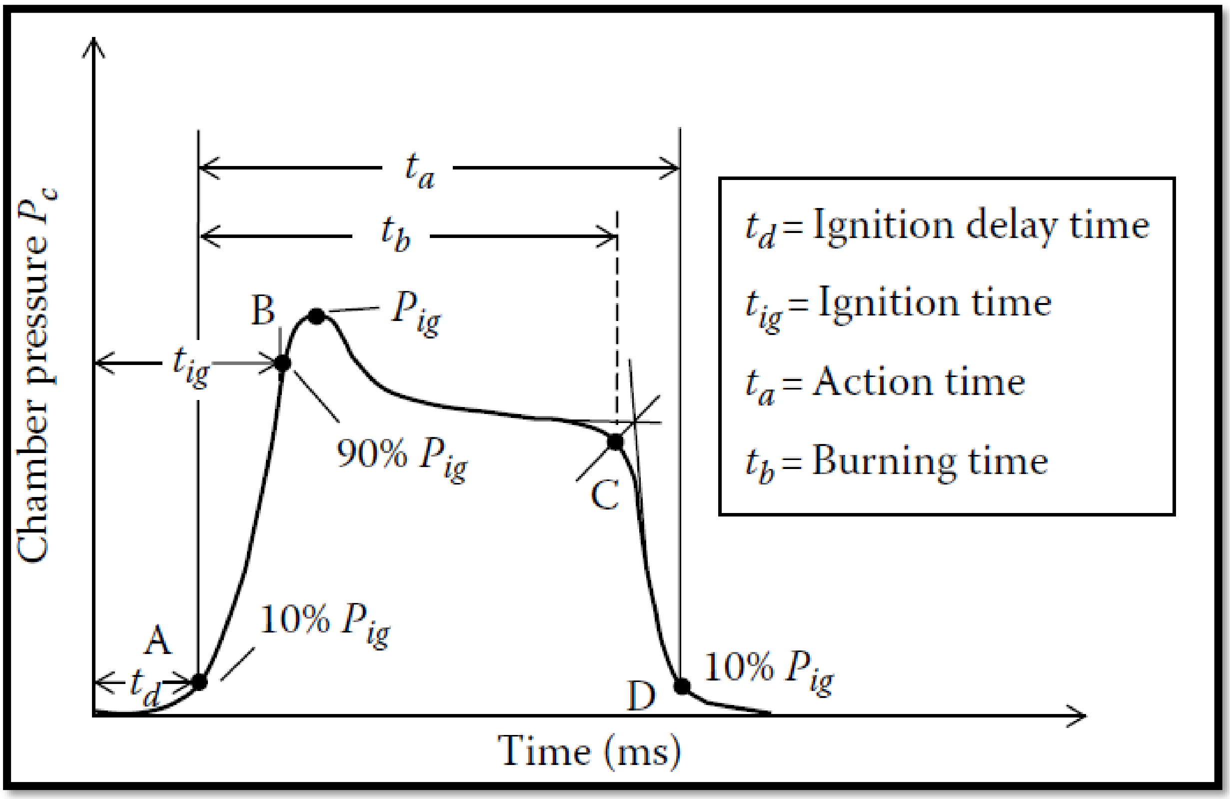
$$(V_{\text{eff}})_{\text{ave}} = \frac{T_t}{\dot{m}_t} = \frac{(1,114 + 2 \times 6,470) \times 1000}{264 + 2 \times 2398} = 2777\,\text{m/s}$$

4. Average specific impulse

$$(I_{Sp})_{\text{ave}} = \frac{(V_{\text{eff}})_{\text{ave}}}{g} = 283\text{ s}$$

Action Time and Burn Time:

A typical variation of chamber pressure with time is shown in following figure



It may be noted that chamber pressure changes very slowly over a small period of time, which is known as ignition delay time t_d. Beyond this time period, chamber pressure rises rapidly due to the generation of burnt gases as a result of combustion of propellants in both igniter and main propellant.

The time duration corresponding to 90% of ignition pressure point B is known as the ignition time t_{ig} .

The intersecting point between the tangent to grain burning and the tangent to the tail-off phase is labelled as point C, which indicates the end of the burning phase.

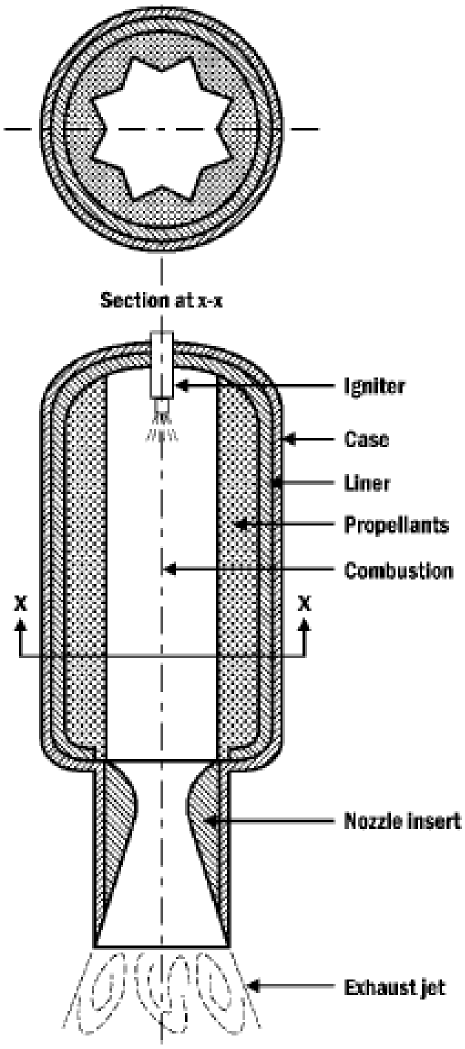
Total burning time is defined as the time between point A and point C. Then rocket can provide thrust beyond the end of propellant burning till the chamber pressure becomes equal to around 10% p_{ig} .

Action time t_a is defined as the time duration between the ignition delay time (point A) and the time corresponding to 10% *Pig* on the tail-off curve point D

Note that the action time is always more than the burning time t_b

Solid Propellant rocket

A simple solid rocket motor consists of a casing, propellant charge (identified as grain), igniter, and nozzle. This grain contains both of the solid fuel and solid oxidizer components combined within a cylindrical combustion chamber or case.



The propellant is casted into the rocket shell having a central cavity of different shapes including star shaped that serves as combustion chamber Thrust-burning time profile depends on this cavity. A liner provided between the case and the propellant protects the case from high temperatures developing inside the propellant layers. An electrical signal is sent to the igniter which creates hot gases that ignite the main propellant grain. Once the flame front is established, combustion is self sustaining. The rate of burning is proportional to the exposed surface area

Solid propellant must have the following features:

1. High chemical energy to generate maximum thrust and specific impulse
2. High density to enable packing a large quantity of propellant in a small volume
3. Low molecular weight thus the exhaust gases achieve high acceleration
4. Easy to ignite
5. Burns steadily at predictive rate
6. Easy to fabricate
7. Smoke-free and nontoxic

Composition of a Solid Propellant:

1. Homogeneous or colloidal propellants

Fuel and oxidizer are contained in the same molecule which decomposes during combustion.
Typical examples are nitroglycerine (NG) and nitrocellulose (NC).

2. Composite or heterogeneous propellants

These are mixtures of oxidizing crystals and a powered fuel (usually aluminium) held together in a binder (synthetic rubber or plastic). Sometimes light metal powders are added to increase the energy of the combustion process and fuel density. They are more stable than homogenous and preferred in long-term stored rockets. However, addition of light metals makes their exhaust toxic and smoky.

3. Composite modified double-based propellants

They are a heterogeneous combination of the double-based homogenous (colloidal) propellants and composite propellants.

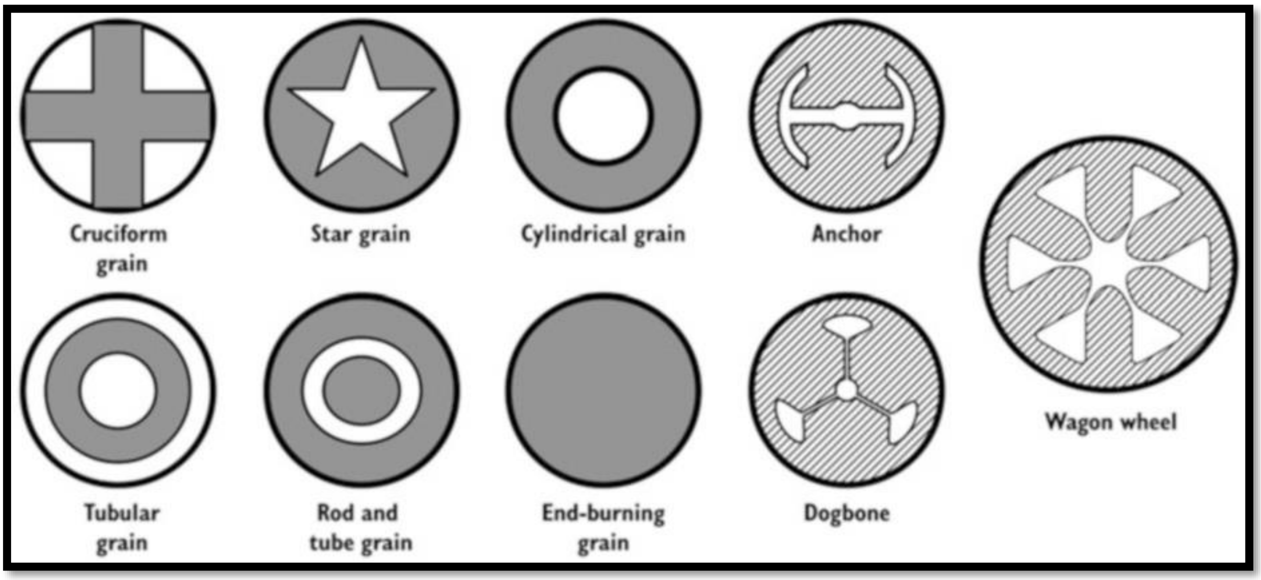
Some Basic definitions are concerning grain:

- **Cylindrical grain:**

A grain in which the internal cross section is constant along the axis regardless of perforation shape.

- **Perforation:**

The central cavity port or flow passage of a propellant grain; its cross section may be cylindrical, tubular, rod, star shape, etc. All have a circular outer boundary due to rocket casing shape.

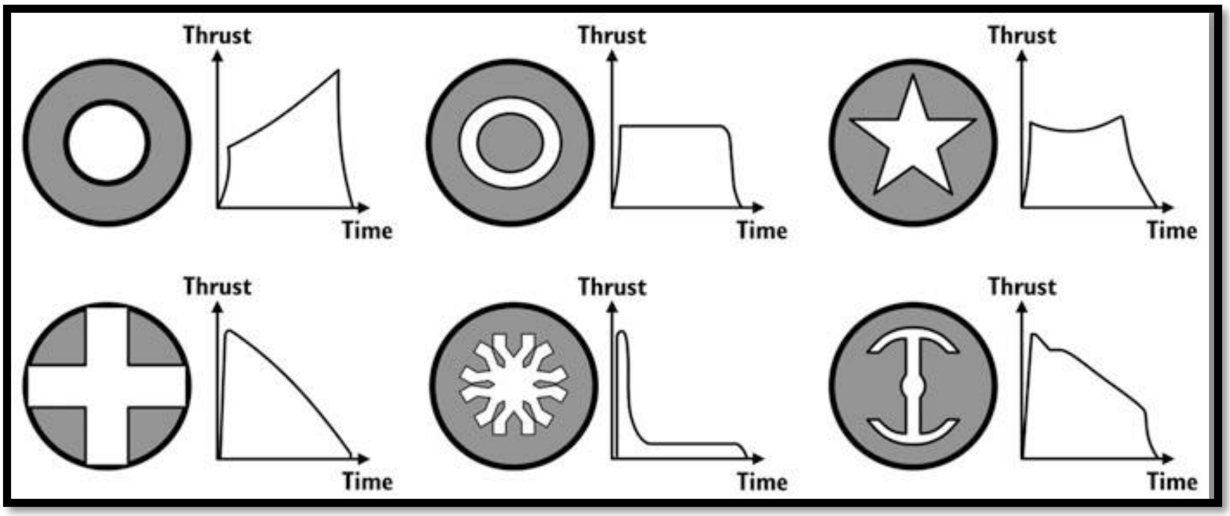


- **End-burning** (or cigarette-burning) grain is the most common and is used.
- **Neutral burning:**

  Motor burn time during which thrust, pressure, and burning surface area remain approximately constant, typically within about +15 %

- **Progressive burning:**

Burn time during which thrust, pressure, and burning surface area increase



Thrust profile for different grain cross sections

- **Regressive burning:**

Burn time during which thrust, pressure, and burning surface area decrease

- **Stoichiometric:**

Mixture of fuel and oxidizer is the correct proportions for complete combustion. Thus the fuel is identified as oxygen balanced.

If the propellant contains insufficient oxygen for complete burning, it is called under oxidized,
and if it contains much oxygen, it is called over-oxidized.

- **Sliver:**

Unburned propellant remaining (or lost, that is, expelled through the nozzle) at the time of web.

The propellant types can be broadly divided into 2 categories:

(1) End burning,

(2) Side (internal) burning

1. End burning grain:

This is the simplest cylindrical grain in which combustion takes place across the transverse cross section of the chamber Hence, it is sometimes also called cigarette burning grain. It is generally used for low-thrust, low performance rocket engines with long burning time as it is limited by a particular cross-sectional burning surface area

2.Side burning grain:

In side burning grain, combustion takes place on the lateral surface of the grain and the end surfaces are protected from burning in contrast to the end burning grain. It may be observed in that burning takes place on the internal surface of the straight cylindrical grain and covers a higher burning surface area compared to end burning grain for the same length and diameter.

The mass generation rate due to the burning of the propellant surface in the combustion chamber can be expressed as follows:

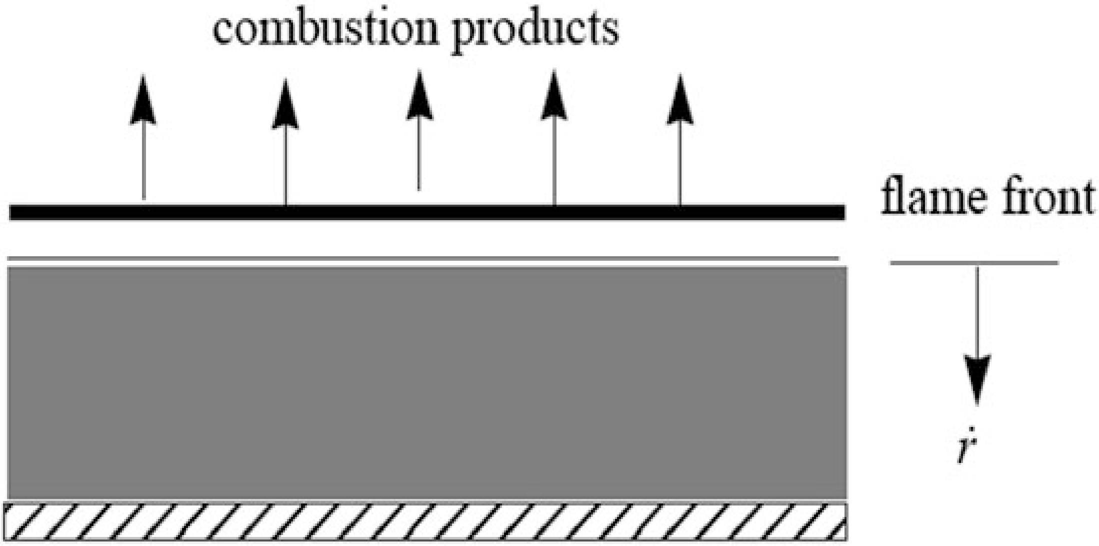
$$\dot{m}_p = \dot{m}_g = \rho_p \dot{r} A_b$$

Where

ρ_p **=It is the density of the propellant**

A_b =It is the burning surface area

$\dot{m}_p = \dot{m}_g$ =The rate of gas generation at the propellant surface

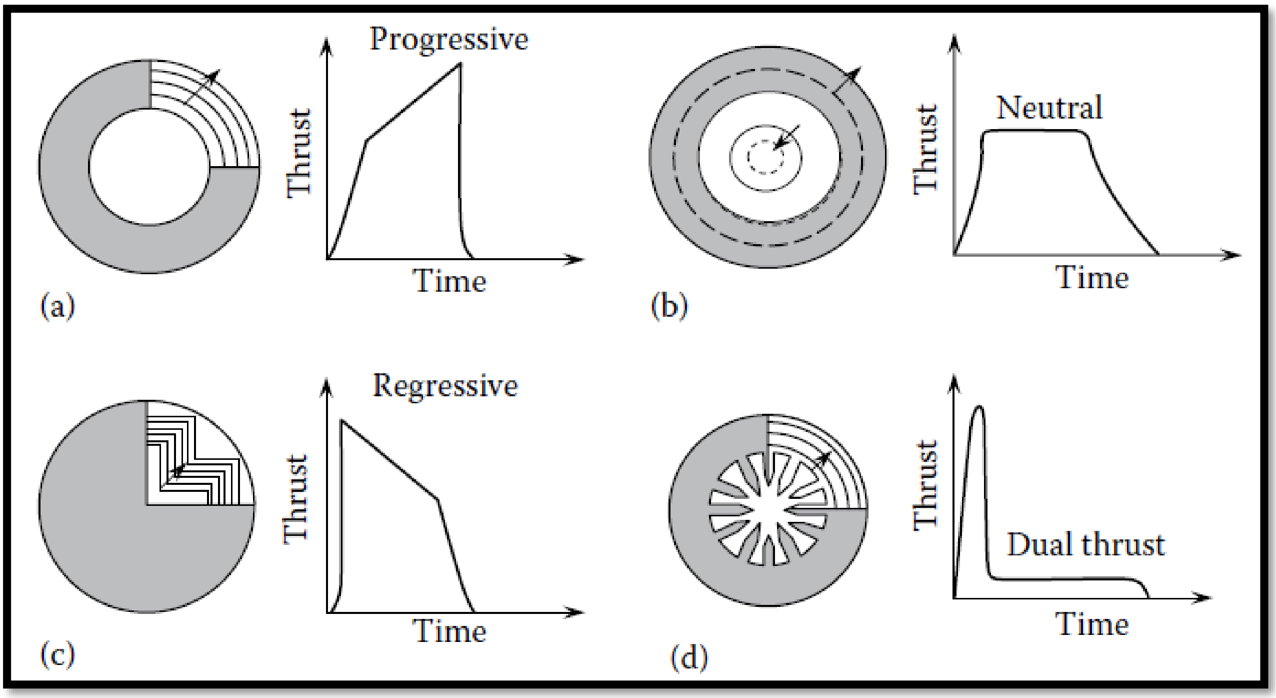


The total burned mass of propellant

$$m_g = \int \dot{m}_g dt = \rho_p \int \dot{r} A_b dt$$

In the case of a cylindrical side burning grain, as the burning surface regresses outward from the inner side of the grain till it reaches the casing wall, the burning surface area

According to chamber pressure/thrust versus time history, propellant grains can be classified further into three types: (1) progressive, (2) neutral, and (3) regressive, as shown in following figure



Burning Rate:

Burning rate is defined as the recession of the propellant surface in a direction perpendicular to this burning surface per unit time.

Burning rate in a full-scale motor depends on :

1. **Propellant composition**
2. **Combustion chamber pressure**
3. **Initial temperature of the solid propellant prior to start**
4. **Combustion gas temperature**
5. **Velocity of the gas flow parallel to the burning surface**
6. **Motor motion (acceleration and spin-induced grain stress)**

At any given initial temperature, the empirical relationship between pressure and burning rate known as Vielle's law may be written as

$$\dot{r} = aP_c^n$$

Where

$\dot{r}$ =The burn rate, usually given in millimetre per second

P_c =The chamber pressure given in MPa

n= The burning rate pressure exponent or combustion index, which is independent of initial grain temperature and describes effect of chamber pressure on the burning rate dependent on the constituent of the propellant

a= An empirical constant influenced by the initial propellant temperature prior to ignition T_p Also it is known as the temperature coefficient which is dimensional quantity. It is expressed by the relation

$$a = \frac{A}{T_1 - T_p}$$

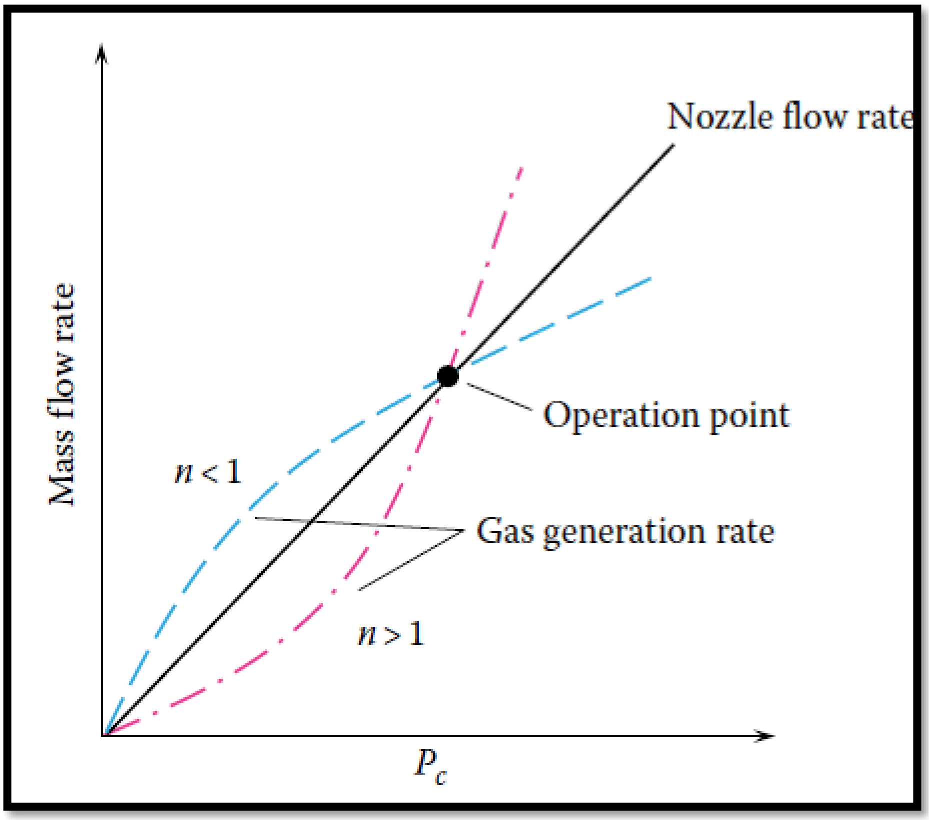
Where both of A and T_1 are empirical constants

Thus rate of burning is given as:

$$\dot{r} = \frac{A}{T_1 - T_p} P_c^n$$

Some features of the exponent (n)

- For stable operation n has values greater than 0 and less than 1.0.
- Most propellants have (n =0.2–0.6).
- If n =0, no change in burning rate over a wide pressure range.
- When the n value is low and comes closer to zero, burning can become unstable and may even extinguish itself.
- As n approaches 1, burning rate and chamber pressure become very sensitive to one another, and disastrous rises in chamber pressure can occur in a few milliseconds.
- Some propellants display a negative (n) which is important for "restartable" motors, burning rate, and chamber pressure.



Rate of mass flow rate in terms of rate of burning

Rate of propellant mass flow is to equal the rate of mass flow through the exhaust nozzle .(mass conservation)

As we know

$$P_c = P_{0e}, T_c = T_{0e}$$

From characteristic Velocity:

$$C^* = P_{0e}\left(\frac{A_{th}}{\dot{m}}\right) = \frac{P_C A^*}{\dot{m}} = \frac{\sqrt{RT_{0e}}}{\sqrt{\gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Or

Mass flow rate of propellant

$$\dot{m}_g = \rho_p A_b \dot{r} = \dot{m} = \frac{A^* P_c}{C^*} = A^* P_c \sqrt{\frac{\gamma}{RT_{0e}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Or

$$P_C = \frac{\rho_p A_b \dot{r} C^*}{A^*} = \frac{\rho_p A_b a P_c^n C^*}{A^*} = \rho_p C^* a P_c^n \frac{A_b}{A^*}$$

Combustion chamber pressure

$$P_C = \left[\rho_p C^* a \left(\frac{A_b}{A^*}\right)\right]^{\frac{1}{1-n}}$$

Hence the ratio between burnt area and throat area:

$$\frac{A_b}{A^*} = \frac{P_C}{\rho_p \dot{r}} \sqrt{\frac{\gamma}{RT_C}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Or

$$\frac{A_b}{A^*} = \frac{P_C^{1-n}}{\rho_p a} \sqrt{\frac{\gamma}{RT_C}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

Problem:

A solid-propellant rocket has the following data
Combustion chamber temperature = 2600 K
Combustion chamber pressure = 18 MPa
Propellant density =1600 kg/m^3
Grain diameter = 10 mm
Exhaust gas constant R = 400J/kgk
Gas-specific heat ratio $\gamma = 1.2$
Vielle's law constants a =4.0, n = 0.6
Burn time =12 s
Exit pressure = 100 kPa

Calculate
(a) The nozzle throat diameter
(b) Characteristic velocity
(c) Optimal thrust coefficient
(d) Thrust force
(e) The mass flow rate and total burnt mass of the propellant
(f) The specific impulse
(g) The total impulse